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Unitary Extension Principle for Nonuniform Wavelet Frames in $L^2(\mathbb{R})$

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Parseval frames have attracted attention of engineers and physicists due to their potential applications in signal processing. In this paper, we study the construction of nonuniform Parseval wavelet frames for the Lebesgue space $L^2(\mathbb{R})$, where the related translation set is not necessary a group. The main purpose of this paper is to prove the unitary extension principle (UEP) and the oblique extension principle (OEP) for the construction of multi-generated nonuniform Parseval wavelet frames for $L^2(\mathbb{R})$. Some examples are also given to illustrate the results.

Key words: Hilbert frame, nonuniform wavelet system, unitary extension principle

Mathematical Subject Classification 2010: 42C40; 42C15; 42C30; 42C05

1. Introduction

Wavelets have been extensively studied over last decades and their role in both pure and applied mathematics is well known. As it is not possible to give a complete list of applications of wavelets, we at least mention some [1,2,7–9,16,18, 19,21,25], see also references therein. Wavelets in $L^2(\mathbb{R})$ are a very efficient tool as they give orthonormal basis for $L^2(\mathbb{R})$ in the form of dilation and translation of a finite number of functions in $L^2(\mathbb{R})$, which is a very simple and convenient form of basis for $L^2(\mathbb{R})$. Gabardo and Nashed [14] considered a generalization of Mallat's classic multiresolution analysis (MRA), which is based on the theory of spectral pairs.

Definition 1.1 ([14, Definition 3.1])**.** Let $N \geq 1$ be a positive integer and r be an odd integer relatively prime to N such that $1 \leq r \leq 2N-1$, an associated nonuniform multiresolution analysis (abbreviated NUMRA) is a collection $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ satisfying the following properties:

(i) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$,

(ii) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$,

(iii) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$,

(iv) $f(x) \in V_j$ if and only if $f(2Nx) \in V_{j+1}$,

(v) there exists a function $\phi \in V_0$, called the *scaling function*, such that the collection $\{\phi(x-\lambda)\}_{\lambda\in\Lambda}$, where $\Lambda = \{0, r/N\}+2\mathbb{Z}$, is a complete orthonormal system for V_0.

Here, the translate set $\Lambda = \{0, r/N\}+2\mathbb{Z}$ may not be a group. One can observe that the standard definition of a one-dimensional multiresolution analysis with dilation factor equal to 2 is a special case of NUMRA given in Definition 1.1. Gabardo and Yu [15] considered the sets of nonuniform wavelets in $L^2(\mathbb{R})$ related to one-dimensional spectral pairs. For fundamental properties of nonuniform wavelets based on spectral pairs, we refer to [14, 15, 23].

Ron and Shen [20] introduced the unitary extension principle for constructing a multi-generated tight wavelet frame for $L^2(\mathbb{R}^d)$ based on a given refinable function. Tight wavelet frames give a more convenient way to represent a function in $L^2(\mathbb{R})$ in comparison with non-tight wavelet frames, as in that case the frame operator is a constant multiple of the identity operator in $L^2(\mathbb{R})$. Christensen and Goh in [6] generalized the unitary extension principle to the locally compact abelian groups. They gave general constructions, based on B-splines on the group itself as well as on the characteristic functions on the dual group. Motivated by the work of Gabardo and Nashed [14] for the construction of nonuniform wavelets and application of frames in applied and pure mathematics, we study nonuniform wavelet frames for the Lebesgue space $L^2(\mathbb{R})$. A notable contribution of the paper is to introduce the unitary extension principle for the construction of multi-generated tight nonuniform wavelet frames of the form

$$\{\Psi_{j,\lambda,\ell}\}_{\substack{j\in\mathbb{Z},\lambda\in\Lambda\\ \ell=1,2,\dots,n}} = \{(2N)^{\frac{j}{2}}\psi_1((2N)^j\gamma-\lambda)\}_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}} \cup\cdots\cup \{(2N)^{\frac{j}{2}}\psi_n((2N)^j\gamma-\lambda)\}_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}}$$

in $L^2(\mathbb{R})$.

1.1. Overview and main results. The paper is organized as follows. In Section 2, we give basic notations, definitions and properties of operators related with nonuniform wavelet frames in $L^2(\mathbb{R})$. The general setup for the nonuniform wavelet frame system in $L^2(\mathbb{R})$ is given in Section 3. Section 4 gives some auxiliary results needed in the rest of the paper. The main results are contained in Section 5. Theorem 5.1 gives the unitary extension principle (UEP) for the construction of multi-generated tight nonuniform wavelet frames for $L^2(\mathbb{R})$. The extended version of UEP (or oblique extension principle) for nonuniform wavelet frames for $L^2(\mathbb{R})$ can be found in Theorem 5.2. Some examples are given in Section 6 to illustrate our results.

1.2. Relation to the existing work and motivation Duffin and Schaeffer [13] introduced the concept of a frame for separable Hilbert spaces, while addressing some difficult problems from the theory of nonharmonic analysis. Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$. The norm induced by the inner product $\langle\cdot,\cdot\rangle$ is given by $\|f\| = \sqrt{\langle f, f\rangle}$, $f \in \mathcal{H}$.

A family $\{f_k\}_{k=1}^{\infty} \subset \mathcal{H}$ is called a *frame* for $\mathcal{H}$ if there exist positive scalars $A_o \leq B_o < \infty$ such that for all $f \in \mathcal{H}$,

$$A_o\|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \leq B_o\|f\|^2. \tag{1.1}$$

The scalars A_o and B_o are called *the lower frame bound* and *the upper frame bound*, respectively. If it is possible to choose $A_o = B_o$, then we say that $\{f_k\}_{k=1}^{\infty}$ is a *A_o-Parseval frame* (or *A_o-tight frame*); and a *Parseval frame* if $A_o = B_o = 1$. If only the upper inequality in (1.1) holds, then we say that $\{f_k\}_{k=1}^{\infty}$ is a *Bessel sequence* with *Bessel bound* B_o. If $\{f_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$, then $S : \mathcal{H} \to \mathcal{H}$, given by $Sf = \sum_{k=1}^{\infty}\langle f, f_k\rangle f_k$, is a frame operator which is bounded linear and invertible on $\mathcal{H}$. This gives the *reconstruction formula* of each member $f \in \mathcal{H}$,

$$f = SS^{-1}f = \sum_{k=1}^{\infty}\langle S^{-1}f, f_k\rangle f_k.$$

Thus, each member of $\mathcal{H}$ has an explicit series expansion which need not be unique. For application of frames in both pure and applied mathematics, we refer to books of Casazza and Kutyniok [3], Christensen [5], Han [16], Heil [17] and Krivoshein, Protasov and Skopina [19]. Nowadays, the theory of iterated function systems, quantum mechanics and wavelets are emerging in important applications in the frame theory, see [12, 22, 24]. A very recent work on discrete frames of translates and discrete wavelet frames and their duals in finite dimensional spaces can be found in [10, 11]. Wavelet frames in $L^2(\mathbb{R})$ are also a very powerful tool for representing functions in $L^2(\mathbb{R})$ as a sum of series of functions which are the dilation and translation of a finite number of functions in $L^2(\mathbb{R})$. It provides us with a convenient tool to expand functions in $L^2(\mathbb{R})$ of a similar type that arise in orthonormal basis, however, the wavelet frame conditions are weaker which makes wavelet frames more flexible. Nonuniform wavelet frames could be used in signal processing, sampling theory, speech recognition and various other areas, where instead of integer shifts nonuniform shifts are needed.

Motivated by the work of Gabardo and Nashed [14] and Gabardo and Yu [15], we study the frame properties of nonuniform wavelets in the Lebesgue space $L^2(\mathbb{R})$. We recall that the extension problems in the frame theory have a long history. It is showed in [4] that the extension problem has a solution in the sense that "any Bessel sequence can be extended to a tight frame by adjoining a suitable family of vectors in the underlying space." Ron and Shen introduced the unitary extension principle for the construction of tight wavelet frames in the Lebesgue space $L^2(\mathbb{R}^d)$. The unitary extension principle allows the construction of tight wavelet frames with compact support of a desired smoothness and a good approximation of functions. In real-life applications, all signals are not obtained from uniform shifts. So there is a natural question regarding analysis and decompositions of these types of signals with a stable mathematical tool. Gabardo and Nashed [14] and Gabardo and Yu [15] filled this gap by the concept of nonuniform multiresolution analysis. In the direction of construction of Parseval frames from nonuniform multiwavelet systems, we develop a general setup and prove the unitary extension principle for the construction of multi-generated nonuniform tight wavelet frames for $L^2(\mathbb{R})$. Ron and Shen [20] gave the unitary extension principle, where the conditions for the construction of multi-generated tight wavelet frames for the Lebesgue space $L^2(\mathbb{R}^d)$ are based on a given refinable function.

2. Preliminaries

As is standard, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{R}$ denote the set of all integers, positive integers and real numbers, respectively. Throughout the paper, $N \in \mathbb{N}$, r being an odd integer relative prime to N such that $1 \leq r \leq 2N-1$ and $\Lambda = \{0, r/N\} + 2\mathbb{Z}$. Notice that the discrete set Λ is not always a group. The support of a function ψ is denoted by $\operatorname{Supp}\psi$ and defined as

$$\operatorname{Supp}\psi := \text{closure of the set } \{x : \psi(x) \neq 0\}.$$

The set of all continuous functions defined on $\mathbb{R}$ with compact support is denoted by $C_c(\mathbb{R})$. The symbol $\overline{z}$ denotes the complex conjugate of a complex number z. The conjugate transpose of a matrix H is denoted by H^*, and the bold number $\mathbf{1}$ denotes the identity matrix. The characteristic function of a set E is denoted by χ_E. The spaces $L^2(\mathbb{R})$ and $L^\infty(\mathbb{R})$ denote the equivalence classes of square-integrable functions and essentially bounded functions on $\mathbb{R}$, respectively. Next, we recall the Parseval identity. Let $\{e_k\}_{k\in\mathbb{Z}}$ be an orthonormal basis for a Hilbert space $\mathcal{H}$. Then, for every f in $\mathcal{H}$, we have

$$\sum_{k\in\mathbb{Z}} |\langle f, e_k\rangle|^2 = \|f\|^2 \quad \text{(Parseval identity)}.$$

For $a, b \in \mathbb{R}$, we consider the following operators on $L^2(\mathbb{R})$:

$$\begin{aligned}
&T_a : L^2(\mathbb{R}) \to L^2(\mathbb{R}), && T_a f(\gamma) = f(\gamma - a) && \text{(Translation by } a\text{)},\\
&E_b : L^2(\mathbb{R}) \to L^2(\mathbb{R}), && E_b f(\gamma) = e^{2\pi i b\gamma} f(\gamma) && \text{(Modulation by } b\text{)},\\
&L : L^2(\mathbb{R}) \to L^2(\mathbb{R}), && L f(\gamma) = \sqrt{2N} f(2N\gamma) && \text{(N-Dilation operator)}.
\end{aligned}$$

The j fold N-dilation, where $j \in \mathbb{Z}$, is given by

$$L^j f(\gamma) = (2N)^{\frac{j}{2}} f((2N)^j \gamma).$$

Definition 2.1. Let $\{\psi_1, \psi_2, \ldots, \psi_n\} \subset L^2(\mathbb{R})$ be a finite set. The family

$$\{L^j T_\lambda \psi_\ell\}_{\substack{j\in\mathbb{Z},\lambda\in\Lambda\\ \ell=1,2,\ldots,n}} = \{(2N)^{\frac{j}{2}} \psi_1((2N)^j\gamma - \lambda)\}_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}} \cup \cdots \cup \{(2N)^{\frac{j}{2}} \psi_n((2N)^j\gamma - \lambda)\}_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}}$$

is called a nonuniform wavelet frame for $L^2(\mathbb{R})$ if there exist finite positive constants A and B such that

$$A\|f\|^2 \leq \sum_{j\in\mathbb{Z}} \sum_{\lambda\in\Lambda} \sum_{\ell=1}^{n} |\langle f, L^j T_\lambda \psi_\ell\rangle|^2 \leq B\|f\|^2 \quad \text{for all } f \in L^2(\mathbb{R}).$$

The Fourier transform of a function $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is denoted by $\mathcal{F}f$ or $\widehat{f}$ and defined as

$$\mathcal{F}f = \widehat{f}(\gamma) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\gamma}\, dx.$$

A function f is said to be bandlimited if its Fourier transform $\widehat{f}$ has a compact support.

For $N \in \mathbb{N}$, $j \in \mathbb{Z}$ and $a \in \mathbb{R}$, by direct calculation, we have the following properties:

(i) $L^j : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is an unitary map,

(ii) $L^j T_a = T_{(2N)^{-j}a} L^j$,

(iii) $\mathcal{F}L^j = L^{-j}\mathcal{F}$,

(iv) $\mathcal{F}T_a = E_{-a}\mathcal{F}$.

The following lemma shows that it is enough to check the Besselness and frame condition on a dense subset of the underlying Hilbert space $\mathcal{H}$.

Lemma 2.2 ([17]). *Let $\{f_k\}_{k\in\mathbb{I}}$, where $\mathbb{I}$ is a countable set, be a sequence of elements in a Hilbert space $\mathcal{H}$.*

(i) *If there exists a constant $B > 0$ such that*

$$\sum_{k\in\mathbb{I}} |\langle f, f_k\rangle|^2 \le B\,\|f\|^2$$

for all f in a dense subset $\mathcal{D}$ of $\mathcal{H}$, then $\{f_k\}_{k\in\mathbb{I}}$ is a Bessel sequence with bound B.

(ii) *If there exist constants $A, B > 0$ such that*

$$A\,\|f\|^2 \le \sum_{k\in\mathbb{I}} |\langle f, f_k\rangle|^2 \le B\,\|f\|^2$$

for all f in a dense subset $\mathcal{D}$ of $\mathcal{H}$, then $\{f_k\}_{k\in\mathbb{I}}$ is a frame for $\mathcal{H}$ with bound B.

3. The nonuniform general setup

In this section, we give a list of assumptions which will be used in the construction of Parseval nonuniform wavelet frames. To be precise, in formulation of the unitary extension principle there is a long list of assumptions, so instead of writing each assumption again and again, we state all assumptions and call them *nonuniform general setup*: Let $\psi_0 \in L^2(\mathbb{R})$ be such that

(i) $\widehat{\psi}_0(2N\gamma) = H_0(\gamma)\widehat{\psi}_0(\gamma),\ H_0(\gamma) \in L^\infty(\mathbb{R})$;

(ii) $\operatorname{Supp}\widehat{\psi}_0(\gamma) \subseteq [0, 1/(4N)]$; and

(iii) $\lim_{\gamma \to 0^+} \widehat{\psi}_0(\gamma) = 1.$

Further, let $H_1, H_2, \ldots, H_n \in L^\infty(\mathbb{R})$, and define $\psi_1, \psi_2, \ldots, \psi_n \in L^2(\mathbb{R})$ such that

$$\widehat{\psi}_\ell(2N\gamma) = H_\ell(\gamma)\widehat{\psi}_0(\gamma), \ \ell = 1, 2, \ldots, n.$$

Let $H(\gamma)$ be an $(n+1) \times 1$ matrix given by

$$H(\gamma) = \begin{bmatrix} H_0(\gamma) \\ H_1(\gamma) \\ \vdots \\ H_n(\gamma) \end{bmatrix}_{(n+1)\times 1}.$$

Then the collection $\{\psi_\ell, H_\ell\}_{\ell=0}^n$ is called a nonuniform general setup.

4. Some auxiliary results

In this section, we give some auxiliary results that will be used in the sequel.

Lemma 4.1. *Assume that*

(i) $\psi_0 \in L^2(\mathbb{R})$, $\lim_{\gamma \to 0^+} \widehat{\psi}_0(\gamma) = 1$ *and* $\operatorname{Supp} \widehat{\psi}_0(\gamma) \subseteq [0, 1/2]$*;*

(ii) $f \in L^2(\mathbb{R})$ *such that* $\widehat{f} \in C_c(\mathbb{R})$*.*

Then, for any $\epsilon > 0$, *there exist* $J \in \mathbb{Z}$ *such that*

$$(1-\epsilon)\|f\|^2 \le \sum_{\lambda \in \Lambda} |\langle f, L^j T_\lambda \psi_0 \rangle|^2 \le (1+\epsilon)\|f\|^2 \text{ for all } j \ge J.$$

Proof. Using $\operatorname{Supp} \widehat{\psi}_0(\gamma) \subseteq [0, 1/2]$ and the Parseval identity on $L^2(0, 1/2)$ with respect to the orthonormal basis $\{\sqrt{2}e^{2\pi i(2m)\gamma}\}_{m\in\mathbb{Z}}$, we compute

$$\begin{aligned}
\sum_{\lambda \in \Lambda} |\langle f, T_\lambda \psi_0 \rangle|^2 &= \sum_{\lambda \in \Lambda} \left| \left\langle \widehat{f}, \widehat{T_\lambda \psi_0} \right\rangle \right|^2 \\
&= \sum_{m\in\mathbb{Z}} \left| \int_0^{\frac{1}{2}} \widehat{f}(\gamma)\overline{\widehat{\psi}_0(\gamma)} e^{2\pi i(2m)\gamma} \, d\gamma \right|^2 \\
&\quad + \sum_{m\in\mathbb{Z}} \left| \int_0^{\frac{1}{2}} \widehat{f}(\gamma)\overline{\widehat{\psi}_0(\gamma)} e^{2\pi i(\frac{r}{N}+2m)\gamma} \, d\gamma \right|^2 \\
&= \int_0^{\frac{1}{2}} \left| \widehat{f}(\gamma)\overline{\widehat{\psi}_0(\gamma)} \right|^2 d\gamma. \qquad (4.1)
\end{aligned}$$

Under the assumption $\widehat{\psi}_0(\gamma) \to 1$ as $\gamma \to 0^+$, it follows that for any $\epsilon > 0$ there exists a sufficiently small positive real number $b = b(\epsilon)$ such that

$$(1-\epsilon)\|\widehat{f}\|^2 \le \sum_{\lambda \in \Lambda} |\langle f, T_\lambda \psi_0 \rangle|^2 \le (1+\epsilon)\|\widehat{f}\|^2,$$

whenever $\operatorname{Supp}\widehat{f} \subseteq [-b,b]$. Now, in order to prove our result, it remains to note that for every bandlimited function f, we have $\langle f, L^j T_\lambda \psi_0\rangle = \langle L^{-j} f, T_\lambda \psi_0\rangle$, $\|L^{-j}f\| = \|f\|$, and $\operatorname{Supp}\widehat{L^{-j}f}$ is supported in $[-b,b]$ for large enough j. This concludes the proof. □

Lemma 4.2. *Suppose that*

(i) $\psi_0 \in L^2(\mathbb{R})$ *satisfies* $\operatorname{Supp}\widehat{\psi_0} \subseteq [0, 1/(4N)]$ *and* $\widehat{\psi}_0(2N\gamma) = H_0(\gamma)\widehat{\psi}_0(\gamma)$, *where* $H_0(\gamma) \in L^\infty(\mathbb{R})$;

(ii) $f \in L^2(\mathbb{R})$ *with* $\widehat{f} \in C_c(\mathbb{R})$, *and* $H_1, H_2, \dots, H_n \in L^\infty(\mathbb{R})$ *such that the* $(n+1)\times 1$ *matrix*

$$H(\gamma) = \begin{bmatrix} H_0(\gamma) \\ H_1(\gamma) \\ \vdots \\ H_n(\gamma) \end{bmatrix}_{(n+1)\times 1}$$

satisfies $H(\gamma)^* H(\gamma) = \mathbf{1}$ *a.e.*;

(iii) $\psi_1, \psi_2, \dots \psi_n \in L^2(\mathbb{R})$ *such that* $\widehat{\psi}_\ell(2N\gamma) = H_\ell(\gamma)\widehat{\psi}_0(\gamma)$, $\ell = 1, 2, \dots n$.

Then

$$\sum_{\ell=0}^{n} \sum_{\lambda\in\Lambda} |\langle f, L^{j-1}T_\lambda \psi_\ell\rangle|^2 = \sum_{\lambda\in\Lambda} |\langle f, L^j T_\lambda \psi_0\rangle|^2.$$

Proof. For any $j \in \mathbb{Z}$ and for any $\ell = 0, 1, \dots n$, we have

$$\begin{aligned}
\langle f, L^{j-1}T_\lambda \psi_\ell\rangle &= \langle L^{-j} f, L^{-1} T_\lambda \psi_\ell\rangle = \langle L^{-j} f, T_{(2N)\lambda} L^{-1}\psi_\ell\rangle = \langle L^j \widehat{f}, E_{-(2N)\lambda} L\widehat{\psi_\ell}\rangle \\
&= \int_{\mathbb{R}} (L^j \widehat{f})(\gamma)\sqrt{2N}\,\overline{\widehat{\psi}_\ell(2N\gamma)}\, e^{2\pi i (2N\lambda)\gamma}\, d\gamma \\
&= \sqrt{2N} \int_{\mathbb{R}} (L^j \widehat{f})(\gamma)\overline{H_\ell(\gamma)}\,\overline{\widehat{\psi}_0(\gamma)}\, e^{2\pi i (2N\lambda)\gamma}\, d\gamma. \qquad (4.2)
\end{aligned}$$

Using $\operatorname{Supp}\widehat{\psi_0} \subseteq [0, 1/(4N)]$, and the Parseval identity on $L^2(0, 1/(4N))$ with respect to the orthonormal basis $\{2\sqrt{N} e^{2\pi i (4Nm)\gamma}\}_{m\in\mathbb{Z}}$, we have

$$\begin{aligned}
\sum_{\lambda\in\Lambda} |\langle f, L^{j-1}T_\lambda \psi_\ell\rangle|^2 &= \sum_{\lambda\in 2\mathbb{Z}} |\langle f, L^{j-1}T_\lambda \psi_\ell\rangle|^2 + \sum_{\lambda\in (\frac{r}{N}+2\mathbb{Z})} |\langle f, L^{j-1}T_\lambda \psi_\ell\rangle|^2 \\
&= \frac{1}{2}\sum_{m\in\mathbb{Z}} \left| \int_0^{\frac{1}{4N}} (L^j \widehat{f})(\gamma)\overline{H_\ell(\gamma)}\,\overline{\widehat{\psi}_0(\gamma)}\, e^{2\pi i (4Nm)\gamma} 2\sqrt{N}\, d\gamma \right|^2 \\
&\quad + \frac{1}{2}\sum_{m\in\mathbb{Z}} \left| \int_0^{\frac{1}{4N}} (L^j \widehat{f})(\gamma)\overline{H_\ell(\gamma)}\,\overline{\widehat{\psi}_0(\gamma)}\, e^{2\pi i (2r)\gamma} e^{2\pi i (4Nm)\gamma} 2\sqrt{N}\, d\gamma \right|^2 \\
&= \frac{1}{2}\int_0^{\frac{1}{4N}} \left| (L^j \widehat{f})(\gamma)\overline{H_\ell(\gamma)}\,\overline{\widehat{\psi}_0(\gamma)} \right|^2 d\gamma
\end{aligned}$$

$$+\frac{1}{2}\int_0^{\frac{1}{4N}}\left|(L^j\widehat{f})(\gamma)\overline{H_\ell(\gamma)}\overline{\widehat{\psi}_0(\gamma)}\right|^2 d\gamma$$
$$=\int_0^{\frac{1}{4N}}\left|(L^j\widehat{f})(\gamma)\overline{H_\ell(\gamma)}\overline{\widehat{\psi}_0(\gamma)}\right|^2 d\gamma.$$

Since $H(\gamma)^*H(\gamma)=\mathbf{1}$ a.e., we have

$$\sum_{\ell=0}^{n}\sum_{\lambda\in\Lambda}|\langle f, L^{j-1}T_\lambda\psi_\ell\rangle|^2=\int_0^{\frac{1}{4N}}\left|(L^j\widehat{f})(\gamma)\overline{\widehat{\psi}_0(\gamma)}\right|^2 d\gamma. \tag{4.3}$$

Also,

$$\begin{aligned}\sum_{\lambda\in\Lambda}|\langle f, L^jT_\lambda\psi_0\rangle|^2 &= \sum_{\lambda\in 2\mathbb{Z}}|\langle f, L^jT_\lambda\psi_0\rangle|^2+\sum_{\lambda\in(\frac{r}{N}+2\mathbb{Z})}|\langle f, L^jT_\lambda\psi_0\rangle|^2\\ &=\sum_{m\in\mathbb{Z}}\left|\int_{\mathbb{R}}(L^j\widehat{f})(\gamma)\overline{\widehat{\psi}_0(\gamma)}e^{2\pi i(2m)\gamma}\,d\gamma\right|^2\\ &\quad+\sum_{m\in\mathbb{Z}}\left|\int_{\mathbb{R}}(L^j\widehat{f})(\gamma)\overline{\widehat{\psi}_0(\gamma)}e^{2\pi i(\frac{r}{N}+2m)\gamma}\,d\gamma\right|^2.\end{aligned} \tag{4.4}$$

Using $\operatorname{Supp}\widehat{\psi}_0\subseteq[0,1/(4N)]\subset[0,1/2]$ and applying the Parseval formula on $L^2(0,1/2)$ with respect to the orthonormal basis $\{\sqrt{2}e^{2\pi i(2m)\gamma}\}_{m\in\mathbb{Z}}$, we compute

$$\begin{aligned}\sum_{\lambda\in\Lambda}|\langle f, L^jT_\lambda\psi_0\rangle|^2 &= \frac{1}{2}\sum_{m\in\mathbb{Z}}\left|\int_0^{\frac{1}{2}}(L^j\widehat{f})(\gamma)\overline{\widehat{\psi}_0(\gamma)}\sqrt{2}e^{2\pi i(2m)\gamma}\,d\gamma\right|^2\\ &\quad+\frac{1}{2}\sum_{m\in\mathbb{Z}}\left|\int_0^{\frac{1}{2}}(L^j\widehat{f})(\gamma)\overline{\widehat{\psi}_0(\gamma)}\sqrt{2}e^{2\pi i(\frac{r}{N}+2m)\gamma}\,d\gamma\right|^2\\ &=\frac{1}{2}\int_0^{\frac{1}{2}}\left|(L^j\widehat{f})(\gamma)\overline{\widehat{\psi}_0(\gamma)}\right|^2 d\gamma+\frac{1}{2}\int_0^{\frac{1}{2}}\left|(L^j\widehat{f})(\gamma)\overline{\widehat{\psi}_0(\gamma)}\right|^2 d\gamma\\ &=\int_0^{\frac{1}{2}}\left|(L^j\widehat{f})(\gamma)\overline{\widehat{\psi}_0(\gamma)}\right|^2 d\gamma=\int_0^{\frac{1}{4N}}\left|(L^j\widehat{f})(\gamma)\overline{\widehat{\psi}_0(\gamma)}\right|^2 d\gamma.\end{aligned} \tag{4.5}$$

The proof now follows from (4.3) and (4.5). □

Lemma 4.3. *Let $\{\psi_\ell, H_\ell\}_{\ell=0}^n$ be a nonuniform general setup, and let $H(\gamma)^*H(\gamma)=\mathbf{1}$. Then the following holds:*

(i) *$\{T_\lambda\psi_0\}_{\lambda\in\Lambda}$ is the Bessel sequence with Bessel bound 1.*

(ii) *For any $f\in L^2(\mathbb{R})$,*

$$\lim_{j\to-\infty}\sum_{\lambda\in\Lambda}|\langle f, L^jT_\lambda\psi_0\rangle|^2=0.$$

Proof. (i) Let $f \in L^2(\mathbb{R})$ be such that $\widehat{f} \in C_c(\mathbb{R})$, and let $\epsilon > 0$ be given. Then, by Lemma 4.1, we can find an integer $j > 0$ such that

$$\sum_{\lambda\in\Lambda} |\langle f, L^j T_\lambda \psi_0\rangle|^2 \leq (1+\epsilon)\|f\|^2. \tag{4.6}$$

Also, by Lemma 4.2, we have

$$\sum_{\lambda\in\Lambda} |\langle f, L^{j-1} T_\lambda \psi_\ell\rangle|^2 \leq \sum_{\lambda\in\Lambda} |\langle f, L^j T_\lambda \psi_0\rangle|^2. \tag{4.7}$$

Applying (4.7) j times and using (4.6), we get

$$\sum_{\lambda\in\Lambda} |\langle f, T_\lambda \psi_0\rangle|^2 \leq \sum_{\lambda\in\Lambda} |\langle f, L^j T_\lambda \psi_0\rangle|^2 \leq (1+\epsilon)\|f\|^2.$$

Since $\epsilon > 0$ is arbitrary, we have

$$\sum_{\lambda\in\Lambda} |\langle f, T_\lambda \psi_0\rangle|^2 \leq \|f\|^2.$$

Because this inequality holds on a dense subset of $L^2(\mathbb{R})$, therefore, by Lemma 2.2, it holds on $L^2(\mathbb{R})$. This proves (i).

(ii) Let $f \in L^2(\mathbb{R})$. Since L^j is an unitary map for all $j \in \mathbb{Z}$, by using (i), the family $\{L^j T_\lambda \psi_0\}_{\lambda\in\Lambda}$ is the Bessel sequence with Bessel bound 1. For any $j \in \mathbb{Z}$ and for any bounded interval $I \subset \mathbb{R}$, we have

$$\begin{aligned}\sum_{\lambda\in\Lambda} |\langle f, L^j T_\lambda \psi_0\rangle|^2 &\leq 2\sum_{\lambda\in\Lambda} |\langle f\chi_I, L^j T_\lambda \psi_0\rangle|^2 + 2\sum_{\lambda\in\Lambda} |\langle f(1-\chi_I), L^j T_\lambda \psi_0\rangle|^2 \\ &\leq 2\sum_{\lambda\in\Lambda} |\langle f\chi_I, L^j T_\lambda \psi_0\rangle|^2 + 2\|f(1-\chi_I)\|^2.\end{aligned}$$

Now, $\|f(1-\chi_I)\|^2 \to 0$ if we choose I to be sufficiently large. Therefore we only need to show

$$\sum_{\lambda\in\Lambda} |\langle f\chi_I, L^j T_\lambda \psi_0\rangle|^2 \to 0 \text{ as } j \to -\infty.$$

Using the Cauchy–Schwarz inequality for integrals, we obtain

$$\begin{aligned}\sum_{\lambda\in\Lambda} |\langle f\chi_I, L^j T_\lambda \psi_0\rangle|^2 &= (2N)^j \sum_{\lambda\in\Lambda} \left| \int_I f(\gamma)\overline{\psi_0((2N)^j\gamma - \lambda)}\, d\gamma \right|^2 \\ &\leq (2N)^j \|f\|^2 \sum_{\lambda\in\Lambda} \int_I \left|\psi_0((2N)^j\gamma - \lambda)\right|^2 d\gamma \\ &= \|f\|^2 \sum_{\lambda\in\Lambda} \int_{(2N)^j I - \lambda} |\psi_0(\gamma)|^2\, d\gamma. \end{aligned} \tag{4.8}$$

Applying the Lebesgue dominated convergence theorem in (4.8), we have

$$\sum_{\lambda\in\Lambda} |\langle f\chi_I, L^j T_\lambda \psi_0\rangle|^2 \to 0 \text{ as } j \to -\infty.$$

Thus (ii) is proved. □

5. The unitary extension principle for nonuniform wavelet frames

We begin this section with the UEP for nonuniform wavelet frames for $L^2(\mathbb{R})$.

Theorem 5.1. *Let $\{\psi_\ell, H_\ell\}_{\ell=0}^{n}$ be a nonuniform general setup and $H(\gamma)^*H(\gamma) = \mathbf{1}$. Then the nonuniform multiwavelet system $\{L^jT_\lambda\psi_\ell\}_{\substack{j\in\mathbb{Z},\lambda\in\Lambda\\ \ell=1,2,\ldots,n}}$ constitutes a Parseval frame for $L^2(\mathbb{R})$.*

Proof. Let $\epsilon > 0$ be given. Consider a function $f \in L^2(\mathbb{R})$ such that $\widehat{f} \in C_c(\mathbb{R})$. By Lemma 4.1, we can choose $J > 0$ such that for all $j \geq J$,

$$(1-\epsilon)\|f\|^2 \leq \sum_{\lambda\in\Lambda} |\langle f, L^jT_\lambda\psi_0\rangle|^2 \leq (1+\epsilon)\|f\|^2. \tag{5.1}$$

Using Lemma 4.2, we have

$$\begin{aligned}\sum_{\lambda\in\Lambda} |\langle f, L^jT_\lambda\psi_0\rangle|^2 &= \sum_{\ell=0}^{n}\sum_{\lambda\in\Lambda} |\langle f, L^{j-1}T_\lambda\psi_\ell\rangle|^2\\ &= \sum_{\lambda\in\Lambda} |\langle f, L^{j-1}T_\lambda\psi_0\rangle|^2 + \sum_{\ell=1}^{n}\sum_{\lambda\in\Lambda} |\langle f, L^{j-1}T_\lambda\psi_\ell\rangle|^2. \end{aligned}\tag{5.2}$$

Applying Lemma 4.2 on $\sum_{\lambda\in\Lambda} |\langle f, L^{j-1}T_\lambda\psi_0\rangle|^2$, we get

$$\sum_{\lambda\in\Lambda} |\langle f, L^{j-1}T_\lambda\psi_0\rangle|^2 = \sum_{\lambda\in\Lambda} |\langle f, L^{j-2}T_\lambda\psi_0\rangle|^2 + \sum_{\ell=1}^{n}\sum_{\lambda\in\Lambda} |\langle f, L^{j-2}T_\lambda\psi_\ell\rangle|^2. \tag{5.3}$$

By (5.2) and (5.3), we have

$$\sum_{\lambda\in\Lambda} |\langle f, L^jT_\lambda\psi_0\rangle|^2 = \sum_{\lambda\in\Lambda} |\langle f, L^{j-2}T_\lambda\psi_0\rangle|^2 + \sum_{\ell=1}^{n}\sum_{\lambda\in\Lambda}\sum_{p=j-2}^{j-1} |\langle f, L^pT_\lambda\psi_\ell\rangle|^2.$$

Repeating the above arguments, for any $m < j$, we have

$$\sum_{\lambda\in\Lambda} |\langle f, L^jT_\lambda\psi_0\rangle|^2 = \sum_{\lambda\in\Lambda} |\langle f, L^{m}T_\lambda\psi_0\rangle|^2 + \sum_{\ell=1}^{n}\sum_{\lambda\in\Lambda}\sum_{p=m}^{j-1} |\langle f, L^pT_\lambda\psi_\ell\rangle|^2. \tag{5.4}$$

It follows from (5.1) and (5.4) that for all $j \geq J$ and for all $m < j$,

$$(1-\epsilon)\|f\|^2 \leq \sum_{\lambda\in\Lambda} |\langle f, L^{m}T_\lambda\psi_0\rangle|^2 + \sum_{\ell=1}^{n}\sum_{\lambda\in\Lambda}\sum_{p=m}^{j-1} |\langle f, L^pT_\lambda\psi_\ell\rangle|^2 \leq (1+\epsilon)\|f\|^2.$$

Letting $m \to -\infty$ in above and using (ii) of Lemma 4.3, we have

$$(1-\epsilon)\|f\|^2 \leq \sum_{\ell=1}^{n}\sum_{\lambda\in\Lambda}\sum_{p=-\infty}^{j-1} |\langle f, L^pT_\lambda\psi_\ell\rangle|^2 \leq (1+\epsilon)\|f\|^2. \tag{5.5}$$

Letting $j \to \infty$ in (5.5), we have

$$(1-\epsilon)\|f\|^2 \le \sum_{\ell=1}^{n} \sum_{\lambda\in\Lambda} \sum_{p=-\infty}^{\infty} |\langle f, L^p T_\lambda \psi_\ell\rangle|^2 \le (1+\epsilon)\|f\|^2.$$

Since $\epsilon > 0$ is arbitrary, we obtain

$$\sum_{\ell=1}^{n} \sum_{\lambda\in\Lambda} \sum_{p\in\mathbb{Z}} |\langle f, L^p T_\lambda \psi_\ell\rangle|^2 = \|f\|^2 \tag{5.6}$$

Now, since (5.6) holds on the dense subset of $L^2(\mathbb{R})$, then, by Lemma 2.2, it will hold on $L^2(\mathbb{R})$, which completes the proof. □

The next theorem gives the generalized (or oblique) extension principle for nonuniform wavelet frames in $L^2(\mathbb{R})$. It gives a more flexible technique to construct nonuniform wavelet frames.

Theorem 5.2. *Let $\{\psi_\ell, H_\ell\}_{\ell=0}^n$ be a nonuniform general setup. Assume that there exists a strictly positive function $\theta \in L^\infty(\mathbb{R})$ for which*

$$\lim_{\gamma\to 0^+} \theta(\gamma) = 1,$$

and

$$\theta(2N\gamma)|H_0(\gamma)|^2 + \sum_{\ell=1}^{n} |H_\ell(\gamma)|^2 = \theta(\gamma).$$

Then $\{L^j T_\lambda \psi_\ell\}_{\substack{j\in\mathbb{Z},\lambda\in\Lambda\\ \ell=1,2,\cdots,n}}$ is a Parseval nonuniform wavelet frame for $L^2(\mathbb{R})$.

Proof. Define $\widetilde{\psi}_0 \in L^2(\mathbb{R})$ such that

$$\widehat{\widetilde{\psi}}_0(\gamma) = \sqrt{\theta(\gamma)}\widehat{\psi}_0(\gamma). \tag{5.7}$$

Define the functions $\widetilde{H}_0, \widetilde{H}_1, \dots, \widetilde{H}_n$ as follows:

$$\widetilde{H}_0(\gamma) = \sqrt{\frac{\theta(2N\gamma)}{\theta(\gamma)}} H_0(\gamma), \quad \widetilde{H}_\ell(\gamma) = \sqrt{\frac{1}{\theta(\gamma)}} H_\ell(\gamma), \qquad \ell = 1, 2, \dots, n.$$

Then we have

$$\begin{aligned}\widehat{\widetilde{\psi}}_0(2N\gamma) &= \sqrt{\theta(2N\gamma)}\widehat{\psi}_0(2N\gamma) = \sqrt{\theta(2N\gamma)} H_0(\gamma)\widehat{\psi_0}(\gamma) \\ &= \sqrt{\theta(2N\gamma)}\left(H_0(\gamma)\frac{\widehat{\widetilde{\psi}}_0(\gamma)}{\sqrt{\theta(\gamma)}}\right) = \sqrt{\frac{\theta(2N\gamma)}{\theta(\gamma)}} H_0(\gamma)\widehat{\widetilde{\psi}}_0(\gamma) \\ &= \widetilde{H}_0(\gamma)\widehat{\widetilde{\psi}}_0(\gamma)\end{aligned} \tag{5.8}$$

and

$$\lim_{\gamma\to 0^+} \widehat{\widetilde{\psi}}_0(\gamma) = \lim_{\gamma\to 0^+} \sqrt{\theta(\gamma)}\widehat{\psi}_0(\gamma) = 1. \tag{5.9}$$

Since $\{\psi_\ell, H_\ell\}_{\ell=0}^n$ is a nonuniform general setup, by (5.7), we have

$$\operatorname{Supp} \widehat{\widetilde{\psi}}_0(\gamma) \subseteq \left[0, \frac{1}{4N}\right] \tag{5.10}$$

and

$$\begin{aligned}\sum_{\ell=0}^{n} |\widetilde{H}_\ell(\gamma)|^2 &= |\widetilde{H}_0(\gamma)|^2 + \sum_{\ell=1}^{n} |\widetilde{H}_\ell(\gamma)|^2 \\ &= \frac{\theta(2N\gamma)}{\theta(\gamma)}|H_0(\gamma)|^2 + \sum_{\ell=1}^{n} \frac{|H_\ell(\gamma)|^2}{\theta(\gamma)} = \frac{1}{\theta(\gamma)}\theta(\gamma) = 1.\end{aligned} \tag{5.11}$$

Thus,

$$\widetilde{H}_\ell(\gamma) \in L^\infty(\mathbb{R}) \text{ for } \ell = 0, 1, \ldots, n. \tag{5.12}$$

Let $\widetilde{\psi}_1, \widetilde{\psi}_2, \ldots, \widetilde{\psi}_n \in L^2(\mathbb{R})$ be such that

$$\widehat{\widetilde{\psi}}_\ell(2N\gamma) = \widetilde{H}_\ell(\gamma)\widehat{\widetilde{\psi}}_0(\gamma), \ \ell = 1, \ldots, n. \tag{5.13}$$

Define

$$\widetilde{H}(\gamma) = \begin{bmatrix} \widetilde{H}_0(\gamma) \\ \widetilde{H}_1(\gamma) \\ \vdots \\ \widetilde{H}_n(\gamma) \end{bmatrix}_{(n+1)\times 1}.$$

Then, by (5.8), (5.9), (5.10) and (5.12), the collection $\{\widetilde{\psi}_\ell, \widetilde{H}_\ell\}_{\ell=0}^n$ is a nonuniform general setup.

Using (5.11), we have

$$\widetilde{H}(\gamma)^*\widetilde{H}(\gamma) = \left[\sum_{\ell=0}^{n} |\widetilde{H}_\ell(\gamma)|^2\right] = \mathbf{1}.$$

Hence, by Theorem 5.1, $\{L^jT_\lambda\widetilde{\psi}_\ell\}_{\substack{j\in\mathbb{Z},\lambda\in\Lambda \\ \ell=1,2,\ldots,n}}$ is a Parseval nonuniform wavelet frame for $L^2(\mathbb{R})$.

Next, we compute

$$\widehat{\psi}_\ell(2N\gamma) = H_\ell(\gamma)\widehat{\psi}_0(\gamma) = \left(\widetilde{H}_\ell(\gamma)\sqrt{\theta(\gamma)}\right)\left(\frac{\widehat{\widetilde{\psi}}_0(\gamma)}{\sqrt{\theta(\gamma)}}\right) = \widetilde{H}_\ell(\gamma)\widehat{\widetilde{\psi}}_0(\gamma) = \widehat{\widetilde{\psi}}_\ell(2N\gamma).$$

This gives $\psi_\ell = \widetilde{\psi}_\ell$. Hence, the system $\{L^jT_\lambda\psi_\ell\}_{\substack{j\in\mathbb{Z},\lambda\in\Lambda \\ \ell=1,2,\cdots,n}}$ is a Parseval nonuniform wavelet frame for $L^2(\mathbb{R})$. □

Remark 5.3. It is worth noticing that, when $\theta = 1$, Theorem 5.1 can be obtained from Theorem 5.2.

Remark 5.4. All the results will also remain true if we replace r/N by an arbitrary $a \in (0,2)$. This is pointed out by the referee.

6. Examples

This section gives some applicative examples of the UEP and its generalized version. The example below illustrates Theorem 5.1.

Example 6.1. Let $N = 2$, $r = 3$, $0 \neq a \in \mathbb{R}$, and $\psi_0 \in L^2(\mathbb{R})$ be such that

$$\widehat{\psi}_0(\gamma) = \frac{\sin(a\gamma)}{a\gamma}\chi_{]0,1/8]}(\gamma).$$

Then

(i) $\lim_{\gamma\to 0^+} \widehat{\psi}_0(\gamma) = 1$;

(ii) $\operatorname{Supp}\widehat{\psi_0} \subseteq \left[0, \frac{1}{8}\right]$; and

(iii) $$\begin{aligned}\widehat{\psi}_0(4\gamma) &= \frac{\sin(4a\gamma)}{4a\gamma}\chi_{]0,1/8]}(4\gamma)\\ &= \frac{4\sin(a\gamma)\cos(a\gamma)\cos(2a\gamma)}{4a\gamma}\chi_{]0,1/32]}(\gamma)\chi_{]0,1/8]}(\gamma) = H_0(\gamma)\widehat{\psi}_0(\gamma),\end{aligned}$$

where $H_0(\gamma) = \cos(a\gamma)\cos(2a\gamma)\chi_{]0,1/32]}(\gamma)$.

Let

$$\begin{aligned}H_1(\gamma) &= \cos(2a\gamma)\sin(a\gamma)\chi_{]0,1/32]}(\gamma),\\ H_2(\gamma) &= \sin(2a\gamma)\chi_{]0,1/32]}(\gamma),\\ H_3(\gamma) &= \chi_{\mathbb{R}\setminus]0,1/32]}(\gamma).\end{aligned}$$

Let $\psi_1, \psi_2, \psi_3 \in L^2(\mathbb{R})$ be such that

$$\widehat{\psi}_\ell(4\gamma) = H_\ell(\gamma)\widehat{\psi}_0(\gamma), \quad \ell = 1, 2, 3.$$

Choose

$$H(\gamma) = \begin{bmatrix} H_0(\gamma)\\ H_1(\gamma)\\ H_2(\gamma)\\ H_3(\gamma)\end{bmatrix}.$$

Then $\{\psi_\ell, H_\ell\}_{\ell=0}^3$ is a nonuniform general setup such that

$$H(\gamma)^* H(\gamma) = \left[|H_0(\gamma)|^2 + |H_1(\gamma)|^2 + |H_2(\gamma)|^2 + |H_3(\gamma)|^2\right] = \mathbf{1}.$$

Hence, by Theorem 5.1, $\{L^j T_\lambda \psi_\ell\}_{j\in\mathbb{Z}, \lambda\in\{0,3/2\}+2\mathbb{Z}, \ell=1,2,3}$ is a nonuniform Parseval wavelet frame for $L^2(\mathbb{R})$.

To conclude the paper, we illustrate Theorem 5.2 with the following example.

Example 6.2. Let $N = 2$, $r = 3$ and $\psi_0 \in L^2(\mathbb{R})$ be such that for any fixed $t \in \mathbb{R}$,

$$\widehat{\psi}_0(\gamma) = e^{it\gamma}\chi_{[0,1/8]}(\gamma).$$

Then

(i) $\lim\limits_{\gamma\to 0^+} \widehat{\psi}_0(\gamma) = 1$;

(ii) $\operatorname{Supp} \widehat{\psi}_0(\gamma) \subseteq \left[0, \frac{1}{8}\right]$; and

(iii) $\psi_0(4\gamma) = e^{4it\gamma}\chi_{[0,1/8]}(4\gamma) = e^{4it\gamma}\chi_{[0,1/32]}(\gamma)\chi_{[0,1/8]}(\gamma) = H_0(\gamma)\widehat{\psi}_0(\gamma)$,

where $H_0(\gamma) = e^{3it\gamma}\chi_{[0,1/32]}(\gamma) \in L^\infty(\mathbb{R})$.

Let $\theta(\gamma) = 1$, and define $H_1(\gamma) = \chi_{\mathbb{R}\setminus[0,1/32]}$. Then the collection $\{\psi_\ell, H_\ell\}_{\ell=0}^{1}$ is a nonuniform general setup such that

$$\theta(4\gamma)|H_0(\gamma)|^2 + |H_1(\gamma)|^2 = \theta(\gamma).$$

Hence, by Theorem 5.2, the nonuniform wavelet system $\{L^j T_\lambda \psi_1\}_{j\in\mathbb{Z},\lambda\in\{0,3/2\}+2\mathbb{Z}}$ is a Parseval frame for $L^2(\mathbb{R})$.

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Принцип унітарного розширення для неоднорідних вейвлет-фреймів в $L^2(\mathbb{R})$

Hari Krishan Malhotra and Lalit Kumar Vashisht

Фрейми Парсеваля привернули увагу інженерів і фізиків завдяки їх потенційному застосуванню в обробці сигналів. У цій роботі ми вивчаємо побудову неоднорідних вейвлет-фреймів Парсеваля для простору Лебега $L^2(\mathbb{R})$, де відповідна множина зсувів не обов'язково має бути групою. Основна мета даної роботи — довести принцип унітарного розширення (ПУР) та принцип косого розширення (ПКР) для побудови мультигенерованих неоднорідних вейвлет-фреймів Парсеваля для $L^2(\mathbb{R})$. Також наведено деякі приклади, що ілюструють результати.

Ключові слова: фрейм Гільберта, неоднорідна вейвлет-система, принцип унітарного розширення