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Biharmonic Hopf Hypersurfaces of Complex Euclidean Space and Odd Dimensional Sphere

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In this paper, biharmonic Hopf hypersurfaces in the complex Euclidean space C^{n+1} and in the odd dimensional sphere S^{2n+1} are considered. We prove that the biharmonic Hopf hypersurfaces in C^{n+1} are minimal. Also, we determine that the Weingarten operator A of a biharmonic pseudo-Hopf hypersurface in the unit sphere S^{2n+1} has exactly two distinct principal curvatures at each point if the gradient of the mean curvature belongs to $D^{\perp}$, and thus is an open part of the Clifford hypersurface $S^{n_1}(1/\sqrt{2}) \times S^{n_2}(1/\sqrt{2})$, where $n_1 + n_2 = 2n$.

Key words: biharmonic hypersurfaces, Hopf hypersurfaces, Chen's conjecture

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1. Introduction

A smooth map $f : (M^m, g) \to (N^n, h)$ between two Riemannian manifolds, M^m being compact, is known to be a harmonic map if it is a critical point of the bienergy functional $E_2(f) = \frac{1}{2}\int_M |\tau(f)|^2 \, d\nu_g$, where $\tau(f) = \operatorname{trace} \nabla df$ denotes the tension field associated to the map f, see [9, 10]. It is known that the vanishing of the tension field characterizes harmonic maps. Also, a new definition of the biharmonic maps associated to the Euler–Lagrange equation was proposed by Jaing [11], that is, f is a biharmonic map if and only if $\tau_2(f) = 0$, where

$$\tau_2(f) = -\Delta\tau(f) - \operatorname{trace} R^N(df(\cdot), \tau(f))df(\cdot). \tag{1.1}$$

The conception and characterization of a biharmonic submanifold based on the use of the mean curvature in the Euclidean space was started independently by B.Y. Chen [5]. Indeed, biharmonic immersions are a special class of biharmonic maps. An isometric immersion $f : (M^m, g) \to (N^n, h)$ is called biharmonic if and only if the mean curvature vector field $\overrightarrow{H}$ of M^m in N^n satisfies equation (1.1), written as

$$0 = \triangle\overrightarrow{H} + \operatorname{trace} R^N(d\varphi(\cdot), \overrightarrow{H})\, d\varphi(\cdot),$$

due to $\tau(f) = mH$. Since each harmonic map is biharmonic because of (1.1), non harmonic biharmonic maps called proper biharmonic maps are of interest. Because in the Euclidean space biharmonic submanifolds and biharmonic immersions coincide, the biharmonicity in Chen's sense can be recovered, i.e., any biharmonic submanifold in the Euclidean space is minimal (harmonic) [6], that is, $\Delta H = 0$.

Nonexistence results for the proper biharmonic maps were obtained in the non-positively curved manifolds. More precisely, in [7], it is proved that in the Euclidean space every biharmonic submanifold with the constant mean curvature is minimal, every hypersurface with at most two distinct principle curvatures in E^m is minimal and a pseudo-umbilical submanifold M^n of $n \neq 4$ is minimal in E^m. Later on, K. Akutagawa and Maeta [2] proved that the biharmonic submanifolds in the Euclidean space are minimal under some circumstances such as completeness. Also, the biharmonic hypersurfaces in the Euclidean 4−space are only minimal ones, see [12]. Recently, a classification of the biharmonic hypersurfaces, depending on the number of distinct principal curvatures, was obtained in [3, 16].

On the other hand, in the positively curved spaces there are several examples of the non minimal biharmonic hypersurfaces. For instance, it was proved that a compact orientable proper biharmonic hypersurface with at most three distinct principal curvatures in the sphere S^{n+1} is either the hypersphere $S^n(\frac{1}{\sqrt{2}})$ or the Clifford hypersurface, see [14]. Furthermore, a complete classification of the proper biharmonic submanifolds in the sphere with the parallel mean curvature vector and parallel Weingarten operator related to the mean curvature vector was obtained in [4].

Actually, the attractiveness of the biharmonic immersion encouraged us to study biharmonic Hopf hypersurfaces in the space form $\overline{M}(c)$. In this paper, we concentrated on the biharmonic Hopf real hypersurfaces M^{2n+1} in C^{n+1} with the natural complex structure and in the unit sphere S^{2n+1}. Here it should be taken into account that the notion of a structural vector field of the hypersurface plays an important role. It is defined by $U = -JN$, where J is the complex structure and N is a local unit normal vector field on the hypersurface M. If U is one of the principle vectors, then hypersurfaces are called Hopf hypersurfaces. Thus, the target of our research is to study the biharmonicity of Hopf CR-hypersurfaces in the complex Euclidean space C^{n+1} and of pseudo-Hopf hypersurfaces in the unit sphere S^{2n+1}.

The paper is organized as follows: in Preliminaries, we recall some essential definitions and give equivalent conditions for a biharmonic hypersurface in the space forms. In Section 3, we prove that the biharmonic Hopf hypersurfaces in C^{n+1} are minimal. Finally, in Section 4, we consider the biharmonic pseudo-Hopf hypersurfaces in the unit sphere S^{2n+1} and obtain that if the Weingarten operator has exactly two distinct principal curvatures at each point, then these hyperspheres are an open part of the Clifford hypersurface $S^{n_1}(1/\sqrt{2}) \times S^{n_2}(1/\sqrt{2})$, where $n_1 + n_2 = 2n$.

2. Preliminaries

Let $x : M^n \to E^{n+1}$ be an isometric immersion of an n-dimensional hypersurface (M^n, g) into the Euclidean space E^{n+1}. Let ∇ and $\overline{\nabla}$ be the Levi-Civita connections on M^n and E^{n+1}, respectively. Suppose that X and Y are tangent vector fields on M^n. Let N be a local unit normal vector field to M^n in E^{n+1}. We recall the Gauss and the Weingarten formulas:

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y),$$
$$\overline{\nabla}_X N = -AX,$$

where A is the Weingarten operator and h is the second fundamental form of M^n. Express the definition of the curvature tensor as

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Furthermore, we have the Gauss and the Codazzi equations

$$R(X, Y)Z = \langle AY, Z\rangle AX - \langle AX, Z\rangle AY,$$
$$(\nabla_X A)Y = (\nabla_Y A)X,$$

respectively, where R is the tensor curvature and X, Y and Z are vectors tangent to M^n. Denote the mean curvature vector by $\overrightarrow{H} = HN$, where H is the mean curvature. The isometric immersion $x : M^n \to E^{n+1}$ is called biharmonic if and only if

$$0 = \triangle\overrightarrow{H} = 2A(\operatorname{grad} H) + nH \operatorname{grad} H + (\triangle H + H \operatorname{trace} A^2),$$

where the Laplacian–Beltrami operator is defined as $\triangle = -\operatorname{trace}\nabla^2$. So, by identifying the tangent and the normal parts of the above equation, we arrive at the necessary and sufficient conditions for M^n to be a biharmonic hypersurface in the Euclidean space E^{n+1}:

$$\triangle H + H \operatorname{trace} A^2 = 0,$$
$$2A(\operatorname{grad} H) + nH \operatorname{grad} H = 0. \tag{2.1}$$

Now we suppose that the ambient space is the complex Euclidean space C^{n+1} which is equipped with the Euclidean metric $\langle\cdot,\cdot\rangle$ viewed as an Hermitian metric $\overline{g}$. We regard J as a map of the tangent bundle $T(C^{n+1})$. There exists a natural basis $\left\{\left(\frac{\partial}{\partial x^1}\right)_x, \left(\frac{\partial}{\partial y^1}\right)_x, \ldots, \left(\frac{\partial}{\partial x^n}\right)_x, \left(\frac{\partial}{\partial y^n}\right)_x\right\}$ for the tangent space $T_x(C^{n+1})$ at a point x, where $(x^1, y^1, \ldots, x^n, y^n)$ are local complex coordinates at x. Put

$$J\left(\frac{\partial}{\partial x^i}\right)_x = \left(\frac{\partial}{\partial y^i}\right)_x,$$
$$J\left(\frac{\partial}{\partial y^i}\right)_x = -\left(\frac{\partial}{\partial x^i}\right)_x$$

for $1 \leq i \leq n$. Then J is called an almost complex structure. A differentiable manifold is called an almost complex manifold if it is equipped with an almost complex structure J satisfying $J^2 = -id$. Consider the complex Euclidean space (C^{n+1}, J). Assume that M is a real hypersurface of C^{n+1}. For any $X \in T_x(M)$, it can be written $JX = FX + u(X)N$, where F is an endomorphism on the tangent bundle $T(C^{n+1})$, u is a one-form and N is a local unit normal vector field on M. Now we consider $U = -JN$, which is called a structural vector field of M in C^{n+1}.

Let N^{2n+1} be an odd dimensional manifold and φ, ξ and η be tensor fields of types $(1,1)$, $(0,1)$ and $(1,0)$ on N^{2n+1}, respectively. For $X \in T(N^{2n+1})$, if the conditions

$$\begin{aligned} \varphi^2(X) &= -X + \eta(X)\xi, \\ \eta(\xi) &= 1, \\ \eta(\varphi(X)) &= 0 \end{aligned}$$

are satisfied, then the triple (φ, ξ, η) is called an almost contact structure and $(N^{2n+1}, \varphi, \xi, \eta)$ is said to be an almost contact manifold. If N^{2n+1} is endowed with a Riemannian metric g, in which

$$\begin{aligned} \eta(X) &= g(\xi, X), \\ g(\varphi X, \varphi Y) &= g(X,Y) - \eta(X)\eta(Y), \\ g(X, \varphi(Y)) &= d\eta(X,Y), \end{aligned} \tag{2.2}$$

where X and Y are in $T(N^{2n+1})$, then $(N^{2n+1}, \varphi, \xi, \eta, g)$ is called a contact metric manifold. Also, the contact metric manifold $(N^{2n+1}, \varphi, \xi, \eta, g)$ is said to be a normal contact metric manifold if $N_\varphi + d\eta \otimes \xi = 0$, where

$$N_\varphi(X,Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X,Y] \tag{2.3}$$

is the Nijenhuis tensor field of φ. In this case, N^{2n+1} is called a Sasakian manifold. The sufficient and necessary condition for a contact metric manifold $(N^{2n+1}, \varphi, \xi, \eta, g)$ to be a Sasakian manifold is well known:

$$(\nabla_X \varphi)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.4}$$

where ∇ is the Levi-Civita connection on N^{2n+1}. In [18], the author proved the following formula for a Sasakian manifold:

$$\nabla_X \xi = -\varphi X. \tag{2.5}$$

Let $(N^{2n+1}, \varphi, \xi, \eta, g)$ be a Sasakian manifold. Then the sectional curvature of a 2-plane spanned by $\{X, \varphi X\}$ is called the φ-sectional curvature, where $X \in T(N^{2n+1})$ is orthogonal to ξ. A Sasakian manifold with the constant φ-sectional curvature c is called a Sasakian space form and denoted by $\overline{N}(c)$. The curvature tensor field of $\overline{N}(c)$ satisfies

$$R(X,Y)Z = -\frac{c-1}{4}\{\eta(Z)[\eta(Y)X - \eta(X)Y] + [g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\xi$$

$$+ g(\varphi X, Z)\varphi Y + 2g(\varphi X, Y)\varphi Z - g(\varphi Y, Z)\varphi X\}$$
$$+ \frac{c+3}{4}\{g(Y, Z)X - g(X, Z)Y\}. \tag{2.6}$$

3. Biharmonic Hopf hypersurfaces in C^{n+1}

In this section, we concentrate on the biharmonic Hopf real hypersurface M^{2n+1} in the complex Euclidean space (C^{n+1}, J). We suppose that the structural vector field U is tangent to M^{2n+1}. Let $T_x(M^{2n+1}) = D \oplus \Re U$, where $D = \{X \in T(M^{2n+1}); u(X) = 0\}$ for all $x \in M^{2n+1}$. Now, by taking the assumptions into account, the following lemmas will be required to reach a suitable frame field for the Weingarten operator.

Lemma 3.1 ([8])**.** *Let U be an eigenvector of the Weingarten operator A corresponding to the eigenvalue α and let X be the eigenvector of A corresponding to the second eigenvalue λ. Then we have*

$$(2\lambda - \alpha)AFX = (2k + \alpha\lambda)FX. \tag{3.1}$$

With respect to the above lemma, we have the result as follows:

Lemma 3.2. *Let M^{2n+1} be a biharmonic Hopf hypersurface of (C^{n+1}, J) with the non-constant mean curvature H. Then we have*

$$\begin{aligned}
&\nabla_{e_1} e_1 = 0,\\
&\nabla_{e_i} e_1 = \sum_{k=1}^{2n+1} \omega_{i1}^k e_k = \omega_{i1}^i e_i,\\
&\nabla_{e_i} F e_j = (\nabla_{e_i} F)e_j + F(\nabla_{e_i} e_j),\\
&\nabla_{e_i} e_{2n+1} = \mu F e_i,\\
&\nabla_{e_{2n+1}} e_{2n+1} = 0,
\end{aligned} \tag{3.2}$$

where ∇ is the Levi-Civita connection on M^{2n+1}, ω_{ij}^k is known as the Cartan coefficient, and $e_{2n+1} = U$.

Proof. It can be obtained from (2.1) that the gradient of the mean curvature H is an eigenvector of the Weingarten operator A corresponding to the eigenvalue $-\frac{(2n+1)}{2}H$. With respect to the proceeding lemma, there is an appropriate orthogonal frame $\{e_1, \ldots, e_n, Fe_1, \ldots, Fe_n, U\}$ such that e_1 can be parallel to $\operatorname{grad} H$. Then the shape operator A of M^{2n+1} takes the form

$$A = \begin{bmatrix}
\lambda_1 & & & & & & \\
& \ddots & & & 0 & & \\
& & \lambda_n & & & & \\
& & & \mu_1 & & & \\
& & & & \ddots & & \\
& & 0 & & & \mu_n & \\
& & & & & & \alpha
\end{bmatrix}, \tag{3.3}$$

where λ_i and μ_i are the eigenvalues corresponding to the eigenvectors e_i and Fe_i, respectively. The eigenvalue α corresponds to the eigenvector U. Taking it into account, the mean curvature is non-constant. Without loss of generality, for $i > n$ we put $e_i = Fe_{i-n}$. Assume that $\operatorname{grad} H$ is given by

$$\operatorname{grad} H = \sum_{i=1}^{2n+1} e_i(H)e_i,$$

so it follows that

$$e_1(H) \neq 0, \quad e_i(H) = 0, \quad i = 2, \ldots, 2n+1, \tag{3.4}$$

because e_1 is parallel to $\operatorname{grad} H$. Also, it can be written

$$\nabla_{e_i} e_j = \sum_{k=1}^{2n+1} \omega_{ij}^k e_k, \tag{3.5}$$

where ω_{ij}^k is the Cartan coefficient. Then, by computing the compatibility conditions, we have

$$\nabla_{e_k}\langle e_i, e_i\rangle = 0, \quad \nabla_{e_k}\langle e_i, e_j\rangle = 0, \tag{3.6}$$

i.e.,

$$\omega_{ki}^i = 0, \quad \omega_{ki}^j + \omega_{ik}^j = 0, \quad i \neq j, \quad i, j, k = 1, \ldots, 2n+1. \tag{3.7}$$

Moreover, it follows from (3.3), (3.5), and (3.6) that the Codazzi equation implies

$$e_i(\lambda_j) = (\lambda_i - \lambda_j)\omega_{ji}^j, \quad (\lambda_i - \lambda_j)\omega_{ki}^j = (\lambda_k - \lambda_j)\omega_{ik}^j \tag{3.8}$$

for distinct $i, j, k = 1, \quad , 2n+1$. From $\lambda_1 = -\frac{2n+1}{2}H$ and (3.4), we get

$$e_1(\lambda_1) \neq 0, \quad e_i(\lambda_1) = 0, \quad i = 2, \ldots, 2n+1, \tag{3.9}$$

and

$$[e_i, e_j]\lambda_1 = 0, \quad 2 \leq i, j \leq 2n+1, \quad i \neq j, \tag{3.10}$$

which imply

$$\omega_{ij}^1 = \omega_{ji}^1 \tag{3.11}$$

for distinct $i, j = 2, \ldots, 2n+1$. It is claimed [15] that $\lambda_j \neq \lambda_1$ for $j = 2, \ldots, 2n+1$. Since $\lambda_j = \lambda_1$ for $j \neq 1$, then, by applying the first term of (3.8) and putting $i = 1$, we have

$$0 = (\lambda_1 - \lambda_j)\omega_{j1}^j = e_1(\lambda_j) = e_1(\lambda_1),$$

which contradicts to (3.9). For $j = 1$ and $k, i \neq 1$, from the second term in (3.8) we get

$$(\lambda_i - \lambda_1)\omega_{ki}^1 = (\lambda_k - \lambda_1)\omega_{ik}^1, \tag{3.12}$$

which together with (3.11) yields

$$\omega_{ij}^1 = 0, \quad i \neq j, \quad i, j = 2, \ldots, 2n+1. \tag{3.13}$$

By combining (3.13) with the second equation of (3.7), it follows that $\omega_{i1}^j = 0$, $i \neq j$, $i, j = 2, \ldots, 2n+1$. Thus, the lemma is proved. □

Now we are ready to prove the principal theorem.

Theorem 3.3. *Let $\psi : M^{2n+1} \to C^{n+1}$ be an isometric immersion of the biharmonic Hopf hypersurface M^{2n+1} in the complex Euclidean space C^{n+1}. Then M^{2n+1} is minimal.*

Proof. We suppose that the mean curvature H on M^{2n+1} is non-constant. Then from the biharmonic condition we have that $\operatorname{grad} H$ is the eigenvector of the Weingarten operator corresponding to the eigenvalue $-\frac{2n+1}{2}H$. We consider the orthogonal frame field $\{e_1, \ldots, e_n, Fe_1, \ldots, Fe_n, U\}$ on the hypersurface, which consists of the eigenvectors of the Weingarten operator corresponding to the eigenvalues satisfying Lemma 3.1. Take into account that e_1 is proportional to $\operatorname{grad} H$. First, we compute the curvature tensor $R(X,Y)Z$ of M^{2n+1}, where X, Y and Z are vectors tangent to M^{2m+1}. Let $X = e_1$ and $Y = Z = e_{2n+1}$. Then, by applying Lemma 3.2, we have

$$R(e_1, e_{2n+1})e_{2n+1} = \lambda_1 \mu_1 e_1. \tag{3.14}$$

On the other hand, the Gauss equation yields

$$R(e_1, e_{2n+1})e_{2n+1} = \langle Ae_{2n+1}, e_{2n+1}\rangle Ae_1 = \alpha\lambda_1 e_1. \tag{3.15}$$

Consequently, from (3.14) and (3.15), we get $\lambda_1(\mu_1 - \alpha) = 0$. Hence, either $\mu_1 = \alpha$ or $\lambda_1 = 0$. However, both choices are impossible because λ_1 is unique and H is non-constant, respectively. Indeed, according to Lemma 3.1, the eigenvalue $\mu_1 = \frac{\lambda_1\alpha}{2\lambda_1 - \alpha}$ corresponding to the eigenvector Fe_1 is equal to α if and only if $\lambda_1 = \alpha$, which is a contradiction. Moreover, $\lambda_1 = 0$ contradicts to the assumption that $\operatorname{grad} H \neq 0$. Therefore, H must be constant. Then, with respect to the biharmonic condition, M^{2n+1} is a minimal hypersurface in the complex Euclidean space C^{n+1}. □

4. Biharmonic pseudo-Hopf hypersurfaces in S^{2n+1}

In this section, we study the biharmonic pseudo-Hopf hypersurfaces M^{2n} in the Sasakian space form S^{2n+1}. Let $x : (M^{2n}, g) \to (S^{2n+1}, \overline{g})$ be an isometric immersion of the biharmonic Riemannian hypersurface M^{2n} in S^{2n+1}. Suppose $\overline{\nabla}$ and ∇ are the Levi-Civita connections on S^{2n+1} and M^{2n}, respectively. Let N be a local unit normal vector field on M^{2n} and $V = -\varphi N$. Then we have

$$T(M^{2n}) = D \oplus D^{\perp} \oplus \Re\xi,$$

where $D = \{X \in T(M^{2n}),\ \eta(X) = 0\}$ is a φ-invariant distribution and $D^{\perp}$ is a one-dimensional subspace spanned by V. Suppose that the Weingarten operator A keeps $\operatorname{span}\{V, \xi\}$, that is, $A(\operatorname{span}\{V, \xi\}) \subseteq \operatorname{span}\{V, \xi\}$ and $AD = D$. A hypersurface M^{2n} is called a pseudo-Hopf hypersurface provided that $\operatorname{span}\{V, \xi\}$ is invariant with respect to the shape operator A, see [1]. By taking equation (2.5) into account, we can have

$$A\xi = -V, \quad \nabla_X V = \varphi AX, \quad \nabla_X \xi = -\varphi X \tag{4.1}$$

for all $X \in D$. Suppose also that $W_1, W_2 \in span\{\xi, V\}$ are the eigenvectors of the Weingarten operator A, $AW_1 = \gamma_1 W_1$ and $AW_2 = \gamma_2 W_2$, such that

$$W_1 = \xi \cos\theta + V \sin\theta, \quad W_2 = -\xi \sin\theta + V \cos\theta \tag{4.2}$$

for some $0 < \theta < \frac{\pi}{2}$. If $AV = \alpha V + \beta\xi$, we have $\beta = -1$ and $\alpha = \frac{\cos 2\theta}{\cos\theta \sin\theta}$.

Lemma 4.1. *Let M^{2n} be a 2n-dimensional pseudo-Hopf hypersurface in the Sasakian space form $(S^{2n+1}, \varphi, \xi, \eta, g)$. If the Weingarten operator A for some $X \in D$ satisfies $AX = \mu X$, then*

$$A\varphi X = \frac{\mu\alpha + 2}{2\mu - \alpha}\varphi X, \tag{4.3}$$

where μ is the eigenvalue corresponding to the eigenvector X.

Proof. Let X and Y in D be the eigenvectors of the shape operator A. Now, by taking the covariant derivative of both sides $AV = \alpha V - \xi$, where $V \in D^\perp$, and applying equation (4.1), we obtain

$$\begin{aligned} \nabla_X AV &= \nabla_X(\alpha V - \xi), \\ (\nabla_X A)V + A(\nabla_X V) &= X(\alpha)V + \alpha\nabla_X V - \nabla_X \xi, \\ (\nabla_X A)V + A(\varphi AX) &= X(\alpha)V + \alpha(\varphi AX) + \varphi X. \end{aligned}$$

Similarly,

$$(\nabla_Y A)V + A(\varphi AY) = Y(\alpha)V + \alpha(\varphi AY) + \varphi Y.$$

Hence,

$$g((\nabla_X A)V, Y) + g(A\varphi AX, Y) = X(\alpha)g(V, Y) + \alpha g(\varphi AX, Y) + g(\varphi X, Y)$$

and

$$g((\nabla_Y A)V, X) + g(A\varphi AY, X) = Y(\alpha)g(V, X) + \alpha g(\varphi AY, X) + g(\varphi Y, X).$$

Then,

$$g((\nabla_X A)Y, V) + g(A\varphi AX, Y) = \alpha g(\varphi AX, Y) + g(\varphi X, Y), \tag{4.4}$$

$$g((\nabla_Y A)X, V) + g(A\varphi AY, X) = \alpha g(\varphi AY, X) + g(\varphi Y, X). \tag{4.5}$$

From (4.4) and (4.5), we have

$$\begin{aligned} &g((\nabla_X A)Y - (\nabla_Y A)X, V) + 2g(A\varphi AX, Y) \\ &\qquad = \alpha g(\varphi AX, Y) + \alpha g(A\varphi X, Y) + 2g(\varphi X, Y). \end{aligned} \tag{4.6}$$

Because M^{2n} is a hypersurface, we have $(\nabla_X A)Y = (\nabla_Y A)X$, and (4.6) yields

$$2g(A\varphi\mu X, Y) = \alpha g(\varphi\mu X, Y) + \alpha g(A\varphi X, Y) + 2g(\varphi X, Y).$$

Then,

$$(2\mu - \alpha)g(A\varphi X, Y) = (\alpha\mu + 2)g(\varphi X, Y). \tag{4.7}$$

Thus the claimed result is obtained. □

Now we are equipped enough to prove the following theorem.

Theorem 4.2. *The biharmonic pseudo-Hopf hypersurfaces in* S^{2n+1} *are an open part of the Clifford hypersurface* $S^{n_1}(\frac{1}{\sqrt{2}}) \times S^{n_2}(\frac{1}{\sqrt{2}})$ *with* $n_1 + n_2 = 2n$ *and* $n_1 \neq n_2$*, wherever* $\operatorname{grad} H$ *belongs to* $D^{\perp}$.

Proof. Actually, the proceeding lemma lets the Weingarten operator A of the pseudo-Hopf hypersurface M^{2n} take the following form with respect to a suitable orthogonal frame field $\{e_1, \dots, e_{n-1}, \varphi e_1, \dots, \varphi e_{n-1}, W_1, W_2\}$:

$$A = \begin{bmatrix} \lambda_1 & & & & & & & \\ & \ddots & & & & 0 & & \\ & & \lambda_{n-1} & & & & & \\ & & & \overline{\lambda}_1 & & & & \\ & & & & \ddots & & & \\ & & & & & \overline{\lambda}_{n-1} & & \\ & & 0 & & & & \gamma_1 & \\ & & & & & & & \gamma_2 \end{bmatrix}, \tag{4.8}$$

where λ_i and $\overline{\lambda}_i$ are the eigenvalues corresponding to the eigenvectors e_i and φe_i, respectively. Everywhere where $i > n-1$, we suppose that $\varphi e_{i+1-n} = e_i$. Take into account that $AW_1 = \gamma_1 W_1$ and $AW_2 = \gamma_2 W_2$. In [1], the authors showed that $\gamma_1 = -\tan\theta$ and $\gamma_2 = \cot\theta$. Consequently, we have $\gamma_1\gamma_2 = -1$. Furthermore, it is deduced that $\operatorname{grad} H$ is the eigenvector of the shape operator A corresponding to the eigenvalue $-nH$, see [5]. So, without loss of generality, we distinguish the following two cases.

Case I: Let e_1 *be a proportion of* $\operatorname{grad} H$. According to this assumption, we can have the corresponding eigenvalue $\lambda_1 = -nH$. Taking into account (3.4)-(3.13), we can write

$$\nabla_{e_i} e_j = \sum_{k=1}^{2n} \omega_{ij}^k e_k \tag{4.9}$$

and

$$\begin{aligned} &\nabla_{e_i}\langle e_j, e_j\rangle = 0, \\ &\nabla_{e_k}\langle e_j, e_i\rangle = \omega_{kj}^i + \omega_{jk}^i, \end{aligned}$$

which imply that

$$\omega_{ij}^j = 0, \quad \omega_{kj}^i + \omega_{jk}^i = 0, \tag{4.10}$$

respectively, for the distinct i, j, where $i, j, k = 1, \dots, 2n$. It is obvious that $[e_i, e_j]\lambda_1 = e_i e_j(\lambda_1) - e_j e_i(\lambda_1) = 0$ where $i, j \neq 1$. Thus,

$$0 = [e_i, e_j]\lambda_1 = (\nabla_{e_i} e_j - \nabla_{e_j} e_i)\lambda_1,$$

which yields

$$\omega_{ij}^1 = \omega_{ji}^1 = 0, \quad i \neq j \neq 1. \tag{4.11}$$

Now, by considering the above equations and applying the appropriate connections, the noticeable result will be obtained in the following. Let $e_{2n} = W_2$. Then, from equation (4.9), we have

$$\nabla_{e_n} e_{2n} = \sum_{k=1}^{2n} \omega_{n2n}^k e_k, \tag{4.12}$$

where $\omega_{n2n}^{2n} = 0$ and $\omega_{n2n}^1 = 0$. On the other hand, with respect to (4.2), we get

$$\begin{aligned}\nabla_{e_n} W_2 &= \nabla_{e_n}(-\xi \sin\theta + V \cos\theta)\\ &= -e_n(\sin\theta)\xi - \sin(\theta)\nabla_{e_n}\xi + e_n(\cos\theta)V + \cos(\theta)\nabla_{e_n}V\\ &= -e_n(\sin\theta)(W_1 \sin\theta + W_2\cos\theta) - \sin\theta(-\varphi e_n)\\ &\quad + e_n(\cos\theta)(W_1\cos\theta - W_2\sin\theta) + \cos\theta(\varphi A e_n)\\ &= (-e_n(\sin\theta)\sin\theta + e_n(\cos\theta)\cos\theta)W_1 - (\sin\theta + \overline{\lambda}_1\cos\theta)e_1\\ &\quad - (e_n(\sin\theta)\cos\theta + e_n(\cos\theta)\sin\theta)W_2.\end{aligned}$$

By the above computation and (4.12), it follows that

$$0 = \omega_{n2n}^1 = \sin\theta + \overline{\lambda}_1\cos\theta.$$

Hence,

$$\overline{\lambda_1} = -\tan\theta. \tag{4.13}$$

In a similar way, by computing $\nabla_{e_n} e_{2n-1}$, where $e_{2n-1} = W_1$, we have

$$0 = \omega_{n2n-1}^1 = \cos\theta - \overline{\lambda}_1\sin\theta,$$

which yields

$$\overline{\lambda}_1 = \cot\theta. \tag{4.14}$$

However, it contradicts to (4.13). Hence, the following result is obtained.

Corollary 4.3. *There is no biharmonic pseudo-Hopf hypersurface M^{2n} of S^{2n+1} with* grad H *belonging to D.*

Case II: Let $W_2 \in D^\perp$ *be collinear to* grad H. We will show that the Weingarten operator A of a pseudo-Hopf biharmonic hypersurface M^{2n} of the Sasakian space form S^{2n+1} has exactly two distinct principle curvatures if grad H is collinear to $W_2 \in D^\perp$.

Due to the above assumption, we have $[e_i, e_j]\gamma_2 = e_i e_j(\gamma_2) - e_j e_i(\gamma_2) = 0$, where $i, j \neq 2n$. So,

$$0 = [e_i, e_j]\gamma_2 = (\nabla_{e_i} e_j - \nabla_{e_j} e_i)\gamma_2, \tag{4.15}$$

which yields that

$$\omega_{ij}^{2n} = \omega_{ji}^{2n}, \quad i \neq j. \tag{4.16}$$

From (3.8), for $j = 2n$ and $k, i \neq 2n$ we have

$$(\lambda_i - \lambda_{2n})\omega_{ki}^{2n} = (\lambda_k - \lambda_{2n})\omega_{ik}^{2n}.$$

Due to $\lambda_i \neq \lambda_k$, we have

$$\omega_{ki}^{2n} = \omega_{ik}^{2n} = 0, \quad i,k \neq 2n. \tag{4.17}$$

Actually, the following computations show us how to get the principal result. Let $e_{2n} = W_2$. Then, from (4.9), we have

$$\nabla_{e_i} e_{2n} = \sum_{k=1}^{2n} \omega_{i2n}^k e_k, \quad i = 1, \dots, n-1, \tag{4.18}$$

where $\omega_{i2n}^n = 0$ for distinct i and n. On the other hand,

$$\begin{aligned}
\nabla_{e_i} W_2 &= \nabla_{e_i}(-\xi \sin\theta + V\cos\theta) \\
&= -e_i(\sin\theta)\xi - \sin(\theta)\nabla_{e_i}\xi + e_i(\cos\theta)V + \cos(\theta)\nabla_{e_i}V \\
&= -e_i(\sin\theta)(W_1\sin\theta + W_2\cos\theta) - \sin\theta(-\varphi e_i) \\
&\quad + e_i(\cos\theta)(W_1\cos\theta - W_2\sin\theta) + \cos\theta(\varphi A e_i) \\
&= (-e_i(\sin\theta)\sin\theta + e_i(\cos\theta)\cos\theta)W_1 + (\sin\theta + \lambda_i\cos\theta)\varphi e_i \\
&\quad - (e_i(\sin\theta)\cos\theta + e_i(\cos\theta)\sin\theta)W_2.
\end{aligned}$$

Hence, from the last relation and (4.18), we obtain that $\lambda_i = -\tan\theta$, where $i = 1, \dots, n-1$. Another useful relation is given by

$$\nabla_{e_j} e_{2n} = \sum_{k=1}^{2n} \omega_{j2n}^k e_k, \quad j = n, \dots, 2n-2, \tag{4.19}$$

where $e_j = \varphi e_{n+1-j}$ is the eigenvector corresponding to the eigenvalue $\overline{\lambda_j}$ with respect to (4.8). Putting $e_{2n} = W_2$, we have

$$\begin{aligned}
\nabla_{e_j} W_2 &= \nabla_{e_j}(-\xi \sin\theta + V\cos\theta) \\
&= -e_j(\sin\theta)\xi - \sin(\theta)\nabla_{e_j}\xi + e_j(\cos\theta)V + \cos(\theta)\nabla_{e_j}V \\
&= -e_j(\sin\theta)(W_1\sin\theta + W_2\cos\theta) - \sin\theta(-\varphi e_j) \\
&\quad + e_j(\cos\theta)(W_1\cos\theta - W_2\sin\theta) + \cos\theta(\varphi A e_j) \\
&= (-e_j(\sin\theta)\sin\theta + e_j(\cos\theta)\cos\theta)W_1 - (\sin\theta + \overline{\lambda}_j\cos\theta)e_i \\
&\quad - (e_j(\sin\theta)\cos\theta + e_j(\cos\theta)\sin\theta)W_2.
\end{aligned}$$

Since $\omega_{j2n}^i = 0$ for the distinct i and j, it follows that $\overline{\lambda}_j = -\tan\theta$, where $j = n, \dots, 2n-2$. Hence, $\lambda_i = \overline{\lambda}_j = -\tan\theta$, where $i = 1, \dots, n-1, j = n, \dots, 2n-1$. Precisely, the above straightforward computation makes it clear that the biharmonic pseudo-Hopf hypersurface M^{2n} in the Sasakian space form S^{2n+1} has two distinct principle curvatures, $-\tan\theta$ and $\cot\theta$, corresponding to the eigenvectors e_i and W_2, respectively, where $i = 1, \dots 2n-1$. Finally, owing to the result from [17], we complete the proof. □

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Бігармонічні поверхні Хопфа в комплексному евклідовому просторі і непарновимірній сфері

Najma Mosadegh and Esmaiel Abedi

У статті розглядаються бігармонічні гіперповерхні Хопфа в комплексному евклідовому просторі C^{n+1} і на непарновимірній сфері S^{2n+1}. Доведено, що бігармонічні гіперповерхні Хопфа в C^{n+1} є мінімальними. Також показано, що якщо градієнт середньої кривини належить до $D^{\perp}$, то оператор Вейнгартена A бігармонічної псевдо-хопфової гіперповерхні на одиничній сфері S^{2n+1} має тільки дві різні головні кривини в кожній точці і, таким чином, гіперповерхня є відкритою частиною гіперповерхні Кліффорда $S^{n_1}(1/\sqrt{2}) \times S^{n_2}(1/\sqrt{2})$, де $n_1 + n_2 = 2n$.

Ключові слова: бігармонічні гіперповерхні, гіперповерхні Хопфа, гіпотеза Чена