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A Nonsingular Action of the Full Symmetric Group Admits an Equivalent Invariant Measure

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Let $\overline{\mathfrak{S}}_\infty$ denote the set of all bijections of natural numbers. Consider an action of $\overline{\mathfrak{S}}_\infty$ on a *measure space* $(X, \mathfrak{M}, \mu)$, where μ is an $\overline{\mathfrak{S}}_\infty$-*quasi-invariant* measure. We prove that there exists an $\overline{\mathfrak{S}}_\infty$-invariant measure equivalent to μ.

Key words: full symmetric group, nonsingular automorphism, Koopman representation, invariant measure.

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1. Introduction

Let $\mathbb{N}$ be the set of all natural numbers and let $\overline{\mathfrak{S}}_\infty$ be the group of all bijections of $\mathbb{N}$. This group is called the *infinite full symmetric group*. Given an element $s \in \overline{\mathfrak{S}}_\infty$, we put $\operatorname{supp} s = \{n \in \mathbb{N} : s(n) \neq n\}$. An element $s \in \overline{\mathfrak{S}}_\infty$ is called finite if $\# \operatorname{supp} s < \infty$. The set of all finite elements form the *infinite symmetric group* denoted by $\mathfrak{S}_\infty$.

Let Aut $(X, \mathfrak{M}, \mu)$ be the set of all *nonsingular* automorphisms of a measure space $(X, \mathfrak{M}, \mu)$. Recall that the automorphism $(X, \mu) \overset{T}{\mapsto} (X, \mu)$ is *nonsingular* if for each measurable $Y \in X$, $\mu(TY) = 0$ if and only if $\mu(Y) = 0$. Throughout the paper we suppose that $\mathfrak{M}$ is the *countably generated* σ-algebra of the measurable subsets of X. A homomorphism α from a group G into Aut $(X, \mathfrak{M}, \mu)$ is called the action of G on $(X, \mathfrak{M}, \mu)$. It is convenient to assume that α is a right action of the group G on X: $X \ni x \overset{\alpha_g}{\mapsto} xg \in X$, $g \in G$. We suppose that

$$\mu\left(\{x \in X : \ x(gh) \neq (xg)h\}\right) = 0$$

for each fixed pair $g, h \in G$ and $Ag^{-1} \in \mathfrak{M}$ for all $A \in \mathfrak{M}$, $g \in G$. Introduce the measure $\mu \circ g$ by setting

$$\mu \circ g(A) = \mu(Ag), A \in \mathfrak{M}.$$

Suppose that the measures μ and $\mu \circ g$ are equivalent (i.e., mutually absolutely continuous) for every $g \in G$. In this case, the measure μ is called G-quasi-invariant. To consider the equivalence class of measures ν, equivalent to μ (the measure class of μ), is the same as to say that the action preserves the measure class of μ. Any measure of the class is transferred to another measure of the same class. Let $\frac{\mathrm{d}\mu\circ g}{\mathrm{d}\mu}$ denote the Radon–Nikodym derivative of $\mu \circ g$ with respect to μ. For more convenience, we put $\rho(g,x) = \sqrt{\frac{\mathrm{d}\mu\circ g}{\mathrm{d}\mu}(x)}$. Then,

$$\int_X (\rho(g,x))^2 f(xg)\,\mathrm{d}\mu = \int_X f(x)\,\mathrm{d}\mu \quad \text{for all } f \in L^1(X,\mu). \tag{1.1}$$

Theorem 1.1. *Let an action of $\overline{\mathfrak{S}}_\infty$ on $(X,\mathfrak{M},\mu)$ be measurable. If the measure μ is $\overline{\mathfrak{S}}_\infty$-quasi-invariant and the σ-algebra $\mathfrak{M}$ is countably generated, then there exists an $\overline{\mathfrak{S}}_\infty$-invariant measure ν (finite or infinite) equivalent to μ.*

2. Outline of the proof of Theorem 1.1

Since the action $X \ni x \mapsto xg \in X$, $g \in \overline{\mathfrak{S}}_\infty$, preserves the measure class μ, we can define the Koopman representation of $\overline{\mathfrak{S}}_\infty$ associated to this action. It is given in the space $L^2(X,\mu)$ by the unitary operators

$$(\mathcal{K}(g)\eta)(x) = \rho(g,x)\eta(xg), \quad \text{where } \eta \in L^2(X,\mu).$$

The separability of σ-algebra $\mathfrak{M}$ implies the separability of unitary group of $L^2(X,\mu)$ in the strong operator topology. Therefore, the homomorphism $\mathcal{K}$ induces the separable topology on $\overline{\mathfrak{S}}_\infty$. But, by [1, Theorem 6.26], $\overline{\mathfrak{S}}_\infty$ has exactly two separable group topologies, namely, the trivial and the standard Polish topologies. The last one is defined by a fundamental system of the neighborhoods $\mathfrak{S}(n,\infty) = \{s \in \overline{\mathfrak{S}}_\infty : \ s(k) = k \text{ for } k = 1,2,\ldots,n\}$ of the identity. Therefore, the representation $\mathcal{K}$ is continuous. It follows that there exists $n \in \mathbb{N} \cup 0$ and a non-zero $\xi \in L^2(X,\mu)$ with the property

$$\mathcal{K}(g)\xi = \xi \quad \text{for all } g \in \mathfrak{S}(n,\infty). \tag{2.1}$$

Set $E = \{x \in X : \xi(x) \neq 0\}$. Using (2.1), we obtain

$$\mu(E\Delta(Eg)) = 0 \quad \text{for all } g \in \mathfrak{S}(n,\infty). \tag{2.2}$$

For $A \subset E$, we define the measure ν by

$$\nu(A) = \int_X \chi_A(x)|\xi(x)|^2 \mathrm{d}\mu.$$

It follows from (2.1) and (2.2) that ν is a $\mathfrak{S}(n,\infty)$-invariant measure on E. This measure can be extended to a $\overline{\mathfrak{S}}_\infty$-invariant measure on X.

3. The properties of continuous representations of $\overline{\mathfrak{S}}_\infty$

To prove Theorem 1.1, we use the general facts about continuous representations of the group $\overline{\mathfrak{S}}_\infty$, which have been well studied by A. Lieberman [2] and G. Olshanski [3, 4]. In this section, we give simple constructions of certain operators and short direct proofs of their properties.

Let $\mathcal{K}$ be a continuous representation of $\overline{\mathfrak{S}}_\infty$ in a Hilbert space $\mathcal{H}$. It follows that for each $\eta \in \mathcal{H}$,

$$\lim_{k\to\infty} \sup_{s\in\mathfrak{S}(k,\infty)} \|\mathcal{K}(s)\eta - \eta\| = 0. \tag{3.1}$$

Set ${}^n\sigma_m = (n+1\ \ n+m+1)(n+2\ \ n+m+2)\cdots(n+m\ \ n+2m)$, where $(k\ j)$ is the permutation of two numbers k, j while all other numbers remain fixed. We need a few auxiliary lemmas.

Lemma 3.1. *The sequence of operators $\{\mathcal{K}({}^n\sigma_m)\}_{m\in\mathbb{N}}$ converges in the weak operator topology to a self-adjoint operator P_n.*

Proof. Let us prove that the sequence $\{\mathcal{K}({}^n\sigma_m)\}_{m\in\mathbb{N}}$ is fundamental in the weak operator topology. Assuming $M > m$, we write ${}^n\sigma_M$ in the form ${}^n\sigma_M = s\,{}^n\sigma_m\,t$, where $s,t \in \mathfrak{S}(n+m,\infty)$. Hence, using (3.1), we have $\lim_{m,M\to\infty} \langle(\mathcal{K}({}^n\sigma_M) - \mathcal{K}({}^n\sigma_m))\,\eta,\zeta\rangle = 0$ for all $\eta,\zeta \in \mathcal{H}$. □

Lemma 3.2. *The operator P_n is a projection.*

Proof. Using lemma 3.1, for any fixed $\eta,\zeta \in \mathcal{H}$, we find sequences $\{m_k\}_{k\in\mathbb{N}}$ and $\{M_k\}_{k\in\mathbb{N}}$ such that $m_{k+1} > m_k$, $M_k > 2m_k$, and

$$\lim_{k\to\infty} \left|\left\langle P_n^2\eta,\zeta\right\rangle - \left\langle \mathcal{K}({}^n\sigma_{M_k})\,\mathcal{K}({}^n\sigma_{m_k})\,\eta,\zeta\right\rangle\right| = 0. \tag{3.2}$$

It should be noticed that ${}^n\sigma_{M_k}\,{}^n\sigma_{m_k} = {}^n\sigma_{m_k}\,s_k$, where $s_k \in \mathfrak{S}(n+m_k,\infty)$. Hence, using (3.1), (3.2), and Lemma 3.1, we have

$$\begin{aligned} 0 &= \lim_{k\to\infty} \left|\left\langle P_n^2\eta,\zeta\right\rangle - \left\langle \mathcal{K}({}^n\sigma_{m_k})\,\mathcal{K}(s_k)\,\eta,\zeta\right\rangle\right| \\ &= \lim_{k\to\infty} \left|\left\langle P_n^2\eta,\zeta\right\rangle - \left\langle \mathcal{K}({}^n\sigma_{m_k})\,\eta,\zeta\right\rangle\right| = \lim_{k\to\infty} \left|\left\langle P_n^2\eta,\zeta\right\rangle - \left\langle P_n\eta,\zeta\right\rangle\right|. \end{aligned}$$ □

Lemma 3.3. *For any $s \in \mathfrak{S}(n,\infty)$, one has $\mathcal{K}(s)P_n = P_n$.*

Proof. Suppose that $m > n$ and $M \geq 2m$. Then $(m\ \ m+1)\,{}^n\sigma_M = {}^n\sigma_M(m+M\ \ m+M+1)$. Hence, applying lemma 3.1 and (3.1), we have

$$\begin{aligned} \langle\mathcal{K}((m\ \ m+1))P_n\eta,\zeta\rangle &= \lim_{M\to\infty} \langle\mathcal{K}((m\ \ m+1))\mathcal{K}({}^n\sigma_M)\eta,\zeta\rangle \\ &= \lim_{M\to\infty} \langle\mathcal{K}({}^n\sigma_M)\mathcal{K}((m+M\ \ m+M+1))\eta,\zeta\rangle \\ &= \lim_{M\to\infty} \langle\mathcal{K}({}^n\sigma_M)\eta,\zeta\rangle \end{aligned}$$

for any η, ζ in $\mathcal{H}$. By lemma 3.1, $\mathcal{K}((m\ \ m+1))P_n = P_n$. Since the transpositions $(m\ \ m+1)$ $(m > n)$ generate the subgroup $\mathfrak{S}(n,\infty)$, the lemma is proved. □

It follows from Lemmas 3.1 and 3.3 that

$$P_n\mathcal{H} = \{\eta \in \mathcal{H} : \mathcal{K}(s)\eta = \eta \text{ for all } s \in \mathfrak{S}(n,\infty)\}. \tag{3.3}$$

Lemma 3.4. *The sequence* $\{\mathcal{K}((k\ \ N))\}_{N\in\mathbb{N}}$ *converges in the weak operator topology to a self-adjoint projection* O_k.

Proof. Using (3.1) and the relation $(k\ \ N_2) = (N_1\ \ N_2)(k\ \ N_1)(k\ \ N_2)$, we deduce that the sequence $\{\mathcal{K}((k\ \ N))\}_{N\in\mathbb{N}}$ is fundamental. Since $(k\ \ N_1)(k\ \ N_2) = (k\ \ N_2)(N_1\ \ N_2)$, the operator O_k is a self-adjoint projection. □

Lemma 3.5. *The projections* P_n *and* O_k *commute:* $P_nO_k = O_kP_n$.

Proof. Since, by Lemma 3.3, $O_kP_n = P_n$ for $k > n$, we suppose that $k \leq n$. By Lemmas 3.1 and 3.4, for any $\eta, \zeta \in \mathcal{H}$, there exists a sequence $\{M_l\}_{l\in\mathbb{N}} \subset \mathbb{N}$ such that $M_{k+1} > M_k$, and

$$\begin{aligned}&\lim_{l\to\infty} \left|\langle P_nO_k\eta, \zeta\rangle - \langle \mathcal{K}\left({}^n\sigma_{M_l}\right) O_k\eta, \zeta\rangle\right| = 0,\\ &\lim_{l\to\infty} \left|\langle O_kP_n\eta, \zeta\rangle - \langle O_k\mathcal{K}\left({}^n\sigma_{M_l}\right) \eta, \zeta\rangle\right| = 0.\end{aligned} \tag{3.4}$$

In the same way, we can find a sequence $\{N_l\}_{l\in\mathbb{N}} \subset \mathbb{N}$ such that $N_{k+1} > N_k > n + 2M_k$, and

$$\lim_{l\to\infty} \left|\langle \mathcal{K}\left({}^n\sigma_{M_l}\right) \mathcal{K}\left(k\ \ N_l\right) \eta, \zeta\rangle - \langle \mathcal{K}\left({}^n\sigma_{M_l}\right) O_k\eta, \zeta\rangle\right| = 0, \tag{3.5}$$

$$\lim_{l\to\infty} \left|\langle \mathcal{K}\left(k\ \ N_l\right) \mathcal{K}\left({}^n\sigma_{M_l}\right) \eta, \zeta\rangle - \langle O_k\mathcal{K}\left({}^n\sigma_{M_l}\right) \eta, \zeta\rangle\right| = 0. \tag{3.6}$$

Now, using (3.4), (3.5) and the relation $(k\ \ N_l)\ {}^n\sigma_{M_l} = {}^n\sigma_{M_l}\ (k\ \ N_l)$, we obtain $P_nO_k = O_kP_n$. □

Lemma 3.6. *Let* $\mathfrak{S}(k, n, \infty)$ *denote the group generated by the transposition* $(k\ \ n+1)$ *and the subgroup* $\mathfrak{S}(n, \infty)$. *Then* O_kP_n *is a self-adjoint projection onto the subspace* $\{\eta \in \mathcal{H} : \mathcal{K}(s)\eta = \eta$ *for all* $s \in \mathfrak{S}(k, n, \infty)\}$. *In particular,* $O_nP_n = P_{n-1}$ *(see*(3.3)*).*

Proof. Due to Lemmas 3.3 and 3.4, the proof follows from the next chain of equalities:

$$\begin{aligned}\langle \mathcal{K}((k\ n+1))O_kP_n\eta, \zeta\rangle &= \lim_{N\to\infty} \langle \mathcal{K}((k\ n+1)(k\ \ N))P_n\eta, \zeta\rangle\\ &= \lim_{N\to\infty} \langle \mathcal{K}((k\ \ N))\mathcal{K}((n+1\ \ N))P_n\eta, \zeta\rangle\\ &= \lim_{N\to\infty} \langle \mathcal{K}((k\ \ N))P_n\eta, \zeta\rangle = \langle O_kP_n\eta, \zeta\rangle\,.\end{aligned}$$ □

Since the representation $\mathcal{K}$ is continuous, then there exists $n \in \mathbb{N}$ such that $P_n \neq 0$. Set $\operatorname{depth}(\mathcal{K}) = \min\{n : P_n \neq 0\}$.

Lemma 3.7. *If* $n = \operatorname{depth}(\mathcal{K})$ *and* $g \notin \mathfrak{S}(n, \infty)$, *then* $P_n\mathcal{K}(g)P_n = 0$.

Proof. Let $k \leq n$ and $g(k) = m > n$. Then $g = (k\ \ m)s$, where $s(m) = m$. Let $\mathbb{S} = \left\{M \in \mathbb{N} : min\left\{M, s^{-1}(M)\right\} > n\right\}$. It is clear that $\#\mathbb{S} = \infty$. Applying Lemmas 3.3 and 3.5, under this condition for $M \in \mathbb{S}$, we have

$$P_n\mathcal{K}(g)P_n = P_n\mathcal{K}((m\ \ M))\mathcal{K}((k\ \ m))\mathcal{K}(s)\mathcal{K}((m)\ \ s^{-1}(M)))P_n$$

$$= P_n\mathcal{K}((m\ \ M))\mathcal{K}((k\ \ m))\mathcal{K}((m\ \ M))\mathcal{K}(s)P_n$$
$$= P_n\mathcal{K}((k\ \ M))\mathcal{K}(s)P_n = P_nO_k\mathcal{K}(s)P_n.$$

But, by (3.3) and Lemma 3.6, taking into account $\mathrm{depth}(\mathcal{K}) = n$, we get

$$\mathcal{K}((k\ \ n))P_nO_k\mathcal{K}((k\ \ n)) = P_nO_n = P_{n-1} = 0.$$

Therefore, $P_n\mathcal{K}(g)P_n = 0$. □

4. The proof of Theorem 1.1

Proof of Theorem 1.1. We follow the notations used in Section 2. Without loss of generality, we may assume that μ is a probability measure. Set $n = \mathrm{depth}(\mathcal{K})$ (see page 49). Recall that we denote by P_n the projection of $L^2(X,\mu)$ onto the subspace $L^2_n = \left\{\eta \in L^2(X,\mu) :\ \mathcal{K}(s)\eta = \eta \text{ for all } s \in \mathfrak{S}(n,\infty)\right\}$. Let the operator $\mathfrak{M}(f)$, where $f \in L^\infty(X,\mu)$, act on $\eta \in L^2(X,\mu)$ as follows:

$$\left(\mathfrak{M}(f)\eta\right)(x) = f(x)\eta(x).$$

Denote by $\mathcal{N}$ the von Neumann algebra generated by $\mathcal{K}(\overline{\mathfrak{S}}_\infty)$ and $\mathfrak{M}(L^\infty(X,\mu))$. Let $\mathbb{S}$ be a subset of $L^2(X,\mu)$, and let $[\mathcal{N}\mathbb{S}]$ be the closure of $\mathcal{N}\mathbb{S}$.

Since $\mathcal{K}$ is continuous (see subsection 2), we have

$$\lim_{k\to\infty} P_k = I. \tag{4.1}$$

If $I - P_l = 0$ for some $l \in \mathbb{N}\cup 0$, then the representation $\mathcal{K}$ is trivial; i. e., $\mathcal{K}(s) = I$ for all $s \in \overline{\mathfrak{S}}_\infty$. Thus we can suppose that $P_l \neq I$ for all $l \in \mathbb{N}\cup 0$.

In the sequel, we will identify the measurable subsets $\mathbb{A}$ and $\mathbb{B}$ if their symmetric difference $\mathbb{A}\Delta\mathbb{B}$ is of measure zero.

Denote by $\widetilde{P}_k$ the orthogonal projection onto the subspace $\left[\mathcal{N}L^2_k\right]$. Since $\widetilde{P}_k$ belongs to the commutant of $\mathcal{N}$, there exists a measurable $\overline{\mathfrak{S}}_\infty$-invariant subset $X_k \subset X$ such that

$$\widetilde{P}_k = \mathfrak{M}(\chi_{X_k}),$$

where χ_{X_k} is the characteristic function of X_k.

Applying (4.1), we obtain

$$X_k \subset X_{k+1} \text{ and } \bigcup_k X_k = X. \tag{4.2}$$

Consider the family of the pairwise orthogonal subspaces $H_0 = L^2_n$, $H_1 = \left(\widetilde{P}_{n+1} - \widetilde{P}_n\right)L^2_{n+1}, \ldots, H_j = \left(\widetilde{P}_{n+j} - \widetilde{P}_{n+j-1}\right)L^2_{n+j}, \ldots$. Using the definitions of $\widetilde{P}_k$ and L^2_k, we conclude from (4.1) that the subspaces $[\mathcal{N}H_k]$ are pairwise orthogonal, and

$$\bigoplus_k [\mathcal{N}H_k] = L^2(X,\mu) \text{ and } P_kH_j = 0 \text{ for all } k < n+j. \tag{4.3}$$

Now we fix the orthonormal basis $\left\{ {}^i\eta_k \right\}_{i=1}^{\dim H_k}$ in H_k. Denote by ${}^i\widetilde{P}_k$ the orthogonal projection onto the subspace $\left[\mathcal{N}\, {}^i\eta_k\right] \subset [\mathcal{N} H_k]$. Then ${}^i\widetilde{P}_k = \mathfrak{M}(\chi_{{}^iX_k})$, where ${}^iX_k$ is a measurable $\overline{\mathfrak{S}}_\infty$-invariant subset of X_k. Since $\left\{{}^i\eta_k\right\}_{i=1}^{\dim H_k}$ is a basis in H_k, we have

$$\bigcup_{i=1}^{\dim H_k} {}^iX_k = X_{n+k} \setminus X_{n+k-1}. \tag{4.4}$$

Define the family $\left\{ {}^iQ_k\right\}_{i=1}^{\dim H_k}$ of the pairwise orthogonal projections as follows:

$${}^1Q_k = {}^1\widetilde{P}_k, \quad {}^2Q_k = {}^2\widetilde{P}_k - {}^2\widetilde{P}_k\, {}^1Q_k, \quad \ldots, \quad {}^lQ_k = {}^l\widetilde{P}_k - {}^l\widetilde{P}_k \sum_{i=1}^{l-1} {}^iQ_k, \quad \ldots$$

It follows from the above discussion that

$${}^i\eta_k \in \bigoplus_{j=1}^{i} \left[\mathcal{N}\, {}^jQ_k\, {}^j\eta_k\right] \quad \text{for all } i = 1, 2, \ldots, \dim H_k. \tag{4.5}$$

Therefore,

$$[\mathcal{N} H_k] = \bigoplus_{j=1}^{\dim H_k} \left[\mathcal{N}\, {}^jQ_k\, {}^j\eta_k\right]. \tag{4.6}$$

As above, ${}^iQ_k = \mathfrak{M}\left(\chi_{,{}^iA_k}\right)$, where $\left\{{}^iA_k\right\}_{i=1}^{\dim H_k}$ is the measurable $\overline{\mathfrak{S}}_\infty$-invariant subset in $X_{n+k} \setminus X_{n+k-1}$ such that ${}^iA_k \cap {}^jA_k = \emptyset$ for different i, j. By (4.4),

$$\sum_{i=1}^{\dim H_k} {}^iQ_k = \widetilde{P}_{n+k} - \widetilde{P}_{n+k-1} \quad \text{and} \quad \bigcup_{i=1}^{\dim H_k} {}^iA_k = X_{n+k} \setminus X_{n+k-1}. \tag{4.7}$$

Denote by ${}^i\mathcal{K}_k$ the restriction of the representation $\mathcal{K}$ to the subspace

$${}^iQ_k L^2(X, \mu) = \left[\mathcal{N}\, {}^i\xi_k\right], \tag{4.8}$$

where ${}^i\xi_k = {}^iQ_k\, {}^i\eta_k$ (see (4.6)). Therefore, if ${}^iQ_k\, {}^i\eta_k \neq 0$, then, using the definitions of H_k, we obtain

$$\operatorname{depth}\left({}^i\mathcal{K}_k\right) = n + k. \tag{4.9}$$

Let us now build an $\overline{\mathfrak{S}}_\infty$-invariant measure ${}^i\nu_k$ on ${}^iA_k$.

Since ${}^i\xi_k = {}^iQ_k\, {}^i\eta_k \in H_k$, we have

$$\left({}^i\mathcal{K}_k(s)\, {}^i\xi_k\right)(x) = \rho(s, x)\, {}^i\xi_k(xs) = {}^i\xi_k(x)$$

for each $s \in \mathfrak{S}(n + k, \infty)$. Therefore, for each $s \in \mathfrak{S}(n + k, \infty)$,

$$\rho(s, x) \left|{}^i\xi_k(xs)\right| = \left|{}^i\xi_k(x)\right|. \tag{4.10}$$

Set ${}^{i}E_k = \left\{x \in X : {}^{i}\xi_k(x) \neq 0\right\}$. It is clear that ${}^{i}E_k \subset {}^{i}A_k$. Since $\mu(\{x \in X : \rho(g,x) = 0\})$, from (4.10), we conclude that for all $s \in \mathfrak{S}(n+k,\infty)$:

$$\mu\left({}^{i}E_k \Delta \left({}^{i}E_k\, s\right)\right) = 0. \tag{4.11}$$

Let us prove that for each $g \notin \mathfrak{S}(n+k,\infty)$,

$$\mu\left(\left({}^{i}E_k\, g\right) \cap {}^{i}E_k\right) = 0. \tag{4.12}$$

Applying (4.9) and Lemma 3.7, we obtain

$$0 = \left\langle {}^{i}\mathcal{K}_k(g) \left|{}^{i}\xi_k\right|, \left|{}^{i}\xi_k\right| \right\rangle = \int_X \rho(g,x) \left|{}^{i}\xi_k(xg)\right| \left|{}^{i}\xi_k(x)\right| \mathrm{d}\mu.$$

Hence, using the equality $\mu\left(\{x \in X : \rho(g,x) = 0\}\right) = 0$, we get that

$$\int_X \left|{}^{i}\xi_k(xg)\right| \left|{}^{i}\xi_k(x)\right| \,\mathrm{d}\mu = 0.$$

Therefore

$$\left|{}^{i}\xi_k(xg)\right| \left|{}^{i}\xi_k(x)\right| = 0$$

holds μ-almost everywhere. Hence (4.12) follows.

Now we define the measure ${}^{i}\mu_k$ on X as follows:

$${}^{i}\mu_k(Y) = \mu(Y \setminus {}^{i}E_k) + \int_{{}^{i}E_k} \chi_{{}_Y}(x) \left|{}^{i}\xi_k(x)\right|^2 \mathrm{d}\mu. \tag{4.13}$$

Assuming that $Y \subset {}^{i}E_k$, $s \in \mathfrak{S}(n+k,\infty)$ and using (1.1), (4.10), (4.11), we obtain

$$\begin{aligned} {}^{i}\mu_k(Ys) &= \int_{{}^{i}E_k} \chi_{{}_{Ys}}(x) \left|{}^{i}\xi_k(x)\right|^2 \mathrm{d}\mu = \int_{{}^{i}E_k} \chi_{{}_Y}(xs^{-1}) \left|{}^{i}\xi_k(x)\right|^2 \mathrm{d}\mu \\ &= \int_{{}^{i}E_k} (\rho(s,x))^2 \chi_{{}_Y}(x) \left|{}^{i}\xi_k(xs)\right|^2 \mathrm{d}\mu \\ &= \int_{{}^{i}E_k} \chi_{{}_Y}(x) \left|{}^{i}\xi_k(x)\right|^2 \mathrm{d}\mu = {}^{i}\mu_k(Y). \end{aligned} \tag{4.14}$$

For the construction of an $\overline{\mathfrak{S}}_\infty$-invariant measure ${}^{i}\nu_k$ on ${}^{i}A_k$, we consider the right coset $H \setminus G$, where $H = \mathfrak{S}(n+k,\infty)$ and $G = \overline{\mathfrak{S}}_\infty$. Since every bijection $s \in G$ can be written as $s = hf$, where $h \in H$ and $f \in \mathfrak{S}_\infty$ is a finite permutation, then there exists a countable full set of the representatives $g_1, g_2, \ldots$ in G of the cosets $H \setminus G$. Define the map $\mathfrak{r} : H \setminus G \mapsto G$ as follows: $\mathfrak{r}(z) = g_j$, if $z = Hg_j$. We will assume that $\mathfrak{r}(H)$ is the identity e of G.

In the sequel, we will need the next useful equality, which follows from (4.8), (4.11) and the definition of ${}^{i}E_k$,

$${}^{i}A_k = \bigcup_{z \in H \setminus G}^{i} E_k\, \mathfrak{r}(z). \tag{4.15}$$

For completeness, we give below a standard algorithm allowing one to extend a finite $\mathfrak{S}(n+k,\infty)$-invariant measure ${}^{i}\mu_k$ on ${}^{i}E_k$ to a σ-finite $\overline{\mathfrak{S}}_\infty$-invariant measure on ${}^{i}\!A_k$.

Take a measurable subset $Y \subset {}^{i}\!A_k$ and define its measure ${}^{i}\nu_k(Y)$ as follows:

$$ {}^{i}\nu_k(Y) = \sum_{z\in H\backslash G} {}^{i}\mu_k\left(\left(Y\cap\left({}^{i}E_k\,\mathfrak{r}(z)\right)\right)(\mathfrak{r}(z))^{-1}\right). \tag{4.16} $$

Let us prove that for all $g \in G$ and $Y \subset {}^{i}A_k$,

$$ {}^{i}\nu_k(Y) = {}^{i}\nu_k(Yg). \tag{4.17} $$

First, we should notice that

$$ \begin{aligned} {}^{i}\nu_k(Yg) &= \sum_{z\in H\backslash G}^{i} \mu_k\left(\left((Yg)\cap\left({}^{i}E_k\mathfrak{r}(z)\right)\right)(\mathfrak{r}(z))^{-1}\right) \\ &= \sum_{z\in H\backslash G}^{i} \mu_k\left(\left(Y\cap\left({}^{i}E_k\mathfrak{r}(z)g^{-1}\right)\right)g(\mathfrak{r}(z))^{-1}\right). \end{aligned} $$

Then, by using (4.11), we get

$$ \begin{aligned} {}^{i}\nu_k(Yg) &= \sum_{z\in H\backslash G}^{i} \mu_k\left(\left(Y\cap\left({}^{i}E_k\mathfrak{r}(zg^{-1})\right)\right)g(\mathfrak{r}(z))^{-1}\right) \\ &= \sum_{z\in H\backslash G}^{i} \mu_k\left(\left(Y\cap\left({}^{i}E_k\mathfrak{r}(zg^{-1})\right)\right)\left(\mathfrak{r}(zg^{-1})\right)^{-1}\mathfrak{r}(zg^{-1})g(\mathfrak{r}(z))^{-1}\right) \\ &= \sum_{z\in H\backslash G}^{i} \mu_k\left(\left(Y\cap\left({}^{i}E_k\mathfrak{r}(z)\right)\right)(\mathfrak{r}(z))^{-1}\mathfrak{r}(z)g(\mathfrak{r}(zg))^{-1}\right), \end{aligned} $$

where $\mathfrak{r}(z)g(\mathfrak{r}(zg))^{-1} \in H = \mathfrak{S}(n+k,\infty)$. Hence, using (4.14), and (4.16), we obtain

$$ {}^{i}\nu_k(Yg) = \sum_{z\in H\backslash G}^{i} \mu_k\left(\left(Y\cap\left({}^{i}E_k\mathfrak{r}(z)\right)\right)(\mathfrak{r}(z))^{-1}\right) = {}^{i}\nu_k(Y). $$

Thus (4.17) is proved.

Now we fix $Y \subset {}^{i}A_k$ such that ${}^{i}\nu_k(Y) = 0$ and prove that $\mu(Y) = 0$.

Indeed, applying (4.16), we have

$$ {}^{i}\mu_k\left(\left(Y\cap\left({}^{i}E_k\,\mathfrak{r}(z)\right)\right)(\mathfrak{r}(z))^{-1}\right) = 0 \quad \text{for all } z \in H\setminus G. $$

It follows from (4.13) that $\mu\left(\left(Y\cap\left({}^{i}E_k\,\mathfrak{r}(z)\right)\right)(\mathfrak{r}(z))^{-1}\right) = 0$ for all $z \in H\setminus G$. Therefore, $\mu\left(\left(Y\cap\left({}^{i}E_k\,\mathfrak{r}(z)\right)\right)\right) = 0$ for all z. Hence, using (4.15), we deduce $\mu(Y) = 0$.

Thus, the restrictions of the measures μ and ${}^{i}\nu_k$ onto ${}^{i}A_k$ are equivalent. Finally, applying (4.7) and (4.2), we conclude that μ is equivalent to the $\overline{\mathfrak{S}}_\infty$-invariant measure $\nu = \sum_{i,k}^{i}\nu_k$. Theorem 1.1 is proved. □

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Існування інваріантної міри для несингулярної дії повної симетричної групи

Nikolay Nessonov

Позначимо через $\overline{\mathfrak{S}}_\infty$ множину всіх бієкцій натуральних чисел. Розглянемо дію $\overline{\mathfrak{S}}_\infty$ на вимірному просторі $(X, \mathfrak{M}, \mu)$, де μ є $\overline{\mathfrak{S}}_\infty$ — *квазиінваріантна* міра. Ми доводимо існування $\overline{\mathfrak{S}}_\infty$-інваріантної міри, яка еквівалентна мірі μ.

Ключові слова: повна симетрична група, несингулярний автоморфізм, купманове зображення, інваріантна міра.