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Implicit Linear Nonhomogeneous Difference Equation in Banach and Locally Convex Spaces

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The paper is dedicated to the 80th anniversary of Anatoliy Georgievich Rutkas

The subjects of this work are the implicit linear difference equations $Ax_{n+1} + Bx_n = g_n$ and $Ax_{n+1} = x_n - f_n, \quad n = 0, 1, 2, \ldots$, where A and B are continuous operators acting in certain locally convex spaces. The existence and uniqueness conditions, along with explicit formulas, are obtained for solutions of these equations. As an application of the general theory produced this way, the equation $Ax_{n+1} = x_n - f_n$ in the space $\mathbb{R}^\infty$ of finite sequences and in the space $\mathbb{R}^M$, where M is an arbitrary set, has been studied.

Key words: difference equation, locally convex space, Banach space, locally nilpotent operator.

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1. Introduction

Let X and Y be locally convex Hausdorff spaces, X^* be a dual to X space endowed with the *strong topology*, i.e., the topology of uniform convergence on every bounded subset of X [6, Chapter IV, §3, Section 1]. Denote by $\mathcal{L}(X,Y)$ the space of linear continuous operators from X into Y. Consider also the space $S(Y) = Y^{\mathbb{N}\cup\{0\}}$ of sequences of elements from Y, endowed by the product topology for a countable collection of copies of Y [6, Chapter I, §1, Section 7]. In what follows, all the spaces are real unless a different assumption is specified explicitly. However, our results are also valid for the associated complex spaces.

Consider the implicit difference equation

$$Ax_{n+1} + Bx_n = g_n, \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

where $A, B \in \mathcal{L}(X,Y)$ and $\{g_n\}_{n=0}^{\infty} \in S(Y)$. In the case of $X = Y$ and $A = I$, the identity operator, equation (1.1), is called *explicit*; otherwise it is *implicit* [5, 23]. With $\operatorname{Ker} A \neq \{0\}$, the implicit difference equation (1.1) is said to be *degenerate* [23, p. 194].

We also consider the following particular case of (1.1):

$$Ax_{n+1} = x_n - f_n, \quad n = 0, 1, 2, \dots \tag{1.2}$$

with $A \in \mathcal{L}(X, X)$ and $\{f_n\}_{n=0}^{\infty} \in S(X)$.

Let the symbol A^* stand for the operator adjoint to A.

A large collection of the existence and uniqueness theorems for the implicit equation (1.1) with a specified initial data was obtained in [3,5,12]. The homogeneous case of (1.1) was studied in [11, Chapter IV, Section 3]. Periodic solutions for explicit difference equations were considered in [7] and for degenerate equations in [23, Subsection 6.4]. Various criteria for the existence and uniqueness of a bounded solution of (1.1) with a bounded sequence $\{f_n\}_{n=0}^{\infty}$ were obtained in [2,13]. Also, the above criteria were obtained in [2,7,22] for difference equations (1.1) considered for $n \in \mathbb{Z}$.

Equation (1.2) can be considered as the linear equation of the second kind

$$\mathcal{T}x + f = x$$

in $S(X)$, where the operator $\mathcal{T} : S(X) \to S(X)$ is defined as follows:

$$(\mathcal{T}x)_n = Ax_{n+1}, \quad n = 0, 1, 2, \dots .$$

In this context, naturally, there arises a problem of describing conditions, under which the only solution of (1.2) is of the form

$$x = (I - \mathcal{T})^{-1} f = \sum_{k=0}^{\infty} \mathcal{T}^k f,$$

i.e.,

$$x_n = \sum_{k=0}^{\infty} A^k f_{n+k}, \quad n = 0, 1, 2, \dots \tag{1.3}$$

(see also Remark 3.4). Furthermore, similar problems are considered for a more general implicit equation (1.1). Some results of that sort were announced in [10] for the case when X and Y are Banach spaces, and a more detailed exposition was presented for the case when $B = -I$ and $X = Y$ is a Fréchet space. Section 3 of this paper provides detailed proofs of the announced results (see Theorem 3.5, Corollary 3.7, Theorem 3.11, and Corollary 3.13). The next result of Section 3 is clarification of general conditions on a locally convex space X and a continuous linear operator A, which guarantee that any solution of (1.2) has the form (1.3) (see Lemma 3.1). We call the associated property of A the *weak local nilpotency* (see Definition 2.4). The relations between this property and the well-known nilpotency and local nilpotency properties are studied in Section 2 (see also Remark 3.3). Furthermore, the relation between these properties and the Bessaga–Pełczyński theorem (see Theorem 2.9) are established in this section.

Section 4 contains studying (1.2) in a dual space produced via an operator in the original space (see equation (4.1)). Assuming that the original space is reflexive and the associated dual is a Fréchet space, a criterion of the existence and uniqueness for a solution of implicit difference equations (4.1) for any sequence of continuous linear functionals is obtained (Theorem 4.1).

Our next step is to consider (1.2) in some special classes of locally convex spaces. It is shown in Sections 5 and 6 that, with X being a sort of space as above, the Banach inverse mapping theorem is valid in $S(X)$. Thus, if (1.2) in these spaces has a unique solution for any sequence $\{f_n\}_{n=0}^{\infty}$, this solution is given by (1.3) (see Theorem 5.1, Corollary 5.2, Theorem 6.1, and Corollary 6.3). The results obtained in these sections are applied, in particular, for studying (1.2) in a purely algebraic situation, when $X = \mathbb{R}^{\infty}$ is the space of finite sequences, which can be identified with the space of polynomials $\mathbb{R}[x]$ (see Example 5.3 and Remark 5.4).

2. Preliminaries

The subjects of this section are certain generalizations of the nilpotency property for continuous linear operators on topological vector spaces. The following lemma shows that the case of nilpotent operator is the simplest in studying (1.2).

Lemma 2.1. *Let $A : X \to X$ be a nilpotent operator with nilpotency index $r+1$ on a vector space X defined everywhere. Then for any sequence $\{f_n\}_{n=0}^{\infty}$ the difference equation* (1.2) *has a unique solution*

$$x_n = \sum_{k=0}^{r} A^k f_{n+k}, \quad n = 0, 1, 2, \dots. \tag{2.1}$$

Proof. Let the nilpotency index of A be equal to $r+1$. Then

$$A^{r+1} = 0. \tag{2.2}$$

A fact that the sequence (2.1) is a solution of equation (1.2) is verified by substituting (2.1) in (1.2) taking into account (2.2). We prove the uniqueness of this solution. For this, we take in (1.2) $f_n = 0,\ n = 0, 1, 2, \dots$ and consider the homogeneous equation

$$Ax_{n+1} = x_n, \quad n = 0, 1, 2, \dots. \tag{2.3}$$

Then, by (2.2), (2.3), we obtain

$$x_n = Ax_{n+1} = A^2 x_{n+2} = \cdots = A^{r+1} x_{n+r+1} = 0, \quad n = 0, 1, 2, \dots.$$

Therefore the homogeneous equation (2.3) has only trivial solution $x_n = 0,\ n = 0, 1, 2, \dots$. The lemma is proved. □

Definition 2.2. Let X be an arbitrary vector space, and $A : X \to X$ be a linear operator defined everywhere. A is called *locally nilpotent* [17, p. 375] if for any $x \in X$ there exists $k = k(x) \in \mathbb{N}$ such that $A^k x = 0$.

Remark 2.3. Note that in a Fréchet space any locally nilpotent operator is nilpotent (see [1, Problem 2.2.6, p. 70], where this was proved for Banach spaces). In more general spaces this fact can fail. For example, the differentiation operator on the space of polynomials with the natural inductive limit topology is locally nilpotent but not nilpotent.

Definition 2.4. An everywhere defined continuous operator $A : X \to X$ on a locally convex Hausdorff space is called *weakly locally nilpotent* if for any $f \in X$ and $\varphi \in X^*$ there exists a positive integer $k = k(f, \varphi)$ such that $\varphi(A^k f) = 0$.

The following two lemmas establish criteria of weak local nilpotency for the operator A, assuming that either X or X^* is a Fréchet space.

Lemma 2.5. *Let X be a Fréchet space. $A \in \mathcal{L}(X, X)$ is a weakly locally nilpotent operator iff A^* is locally nilpotent.*

Proof. Sufficiency. If A^* is a locally nilpotent operator, then for any $\varphi \in X^*$ there exists a number $k = k(\varphi) \in \mathbb{N}$ such that for any $f \in X$ the following equalities are fulfilled: $0 = (A^*)^k\varphi(f) = \varphi(A^k f)$. Therefore the operator A is weakly locally nilpotent.

Necessity. We fix an arbitrary functional $\varphi \in X^*$. For any $k \in \mathbb{N}$, we consider the closed set $X_k = \{f \in X : \varphi(A^k f) = 0\}$. Since the operator A is weakly locally nilpotent, we have that $X = \bigcup_{k=1}^{\infty} X_k$. Since X is a complete metrizable space, it follows from the Baire theorem that X is a second category set. Consequently, for some $k = k(\varphi) \in \mathbb{N}$, the set X_k contains a ball $B_r(x_0) = \{x \in X : d_X(x, x_0) \leq r\}$, where $d_X(\cdot, \cdot)$ is a distance in the Fréchet space X. Therefore $\varphi(A^k x) = 0$ for any $x \in B_r(x_0)$. Taking into account that a neighborhood of zero is an absorbent set, we conclude that $\varphi(A^k x) = 0$ for any $x \in X$. Thus, $(A^*)^k\varphi = 0$, i.e., the operator $A^* : X^* \to X^*$ is locally nilpotent. The lemma is proved. □

Lemma 2.6. *Let X be an arbitrary locally convex Hausdorff space such that its dual X^* is a Fréchet space with respect to the strong topology. An everywhere defined operator $A : X \to X$ is weakly locally nilpotent iff A is locally nilpotent.*

Proof. Sufficiency is obvious from the lemma assertion.

Necessity is to be proved. Let $f \in X$. For any $k \in \mathbb{N}$, we consider a closed subset $\tilde{X}_k = \{\varphi \in X^* : \varphi(A^k f) = 0\}$ of the Fréchet space X^*. Since the operator A is weakly locally nilpotent, we have $X^* = \bigcup_{k=1}^{\infty} \tilde{X}_k$. Since X^* is a complete metrizable space, it follows from the Baire theorem that X^* is a second category set. Consequently, for some $k = k(f) \in \mathbb{N}$, the set X_k contains a ball $B_r(\varphi_0) = \{\varphi \in X^* : d_{X^*}(\varphi, \varphi_0) \leq r\}$, where $d_{X^*}(\cdot, \cdot)$ is a distance in the Fréchet space X^*. Therefore, $\varphi(A^k f) = 0$ for any $\varphi \in B_r(\varphi_0)$. Taking into account that a neighborhood of zero is an absorbent set, we conclude that $\varphi(A^k f) = 0$ for any $\varphi \in X^*$, therefore by the Hahn–Banach theorem, $A^k f = 0$. Thus, the operator A is locally nilpotent. The lemma is proved. □

Remark 2.7. Suppose that both X and X^* are Fréchet spaces. Then, by Proposition 15 [20, Chapter 6, §3], X is a Banach space, and in this case the weak local nilpotency of an operator A is equivalent to its nilpotency by Remark 2.3 and Lemma 2.6.

Now we show that a weakly locally nilpotent operator on a Fréchet space may appear to be non-locally nilpotent.

Example 2.8. Let $X = s = S(\mathbb{R})$ be the space of all sequences of real numbers with the topology of the coordinate-wise convergence, A be the right shift i.e., $Au = (0, u_0, u_1, u_2, \dots)$ for $u = (u_0, u_1, u_2, \dots) \in s$. The operator A is defined and continuous on X, but it is not locally nilpotent. Let us show that it is weakly locally nilpotent. The general form of a linear continuous functional $\varphi \in s^*$ is given by

$$\varphi(x) = \sum_{k=0}^{m} a_k x_k, \quad x = (x_0, x_1, x_2, \dots) \in s$$

[15, p. 284]. Then, for $n > m$, one has $\varphi(A^n x) = 0$ for any $x \in s$. Therefore the operator A is weakly locally nilpotent.

The following Theorem shows that all the examples of weakly locally nilpotent but not nilpotent operators in a Fréchet space are somehow related to the space s.

Theorem 2.9. *Let X be a Fréchet space. An arbitrary continuous weakly locally nilpotent operator on X is nilpotent iff X does not contain a closed subspace isomorphic to the space s.*

Proof. Necessity. Let X contain a closed subspace isomorphic to s. Since there is no norm, which is continuous with respect to the topology of the space s (see, for example, [4, Corollary 1]), then there is no such a norm on X. By the Bessaga–Pelczyński theorem [4, Theorem 2], the following decomposition of the space X into a direct sum is fulfilled: $X = X_1 \oplus X_2$, where X_1, X_2 are closed spaces and X_1 is isomorphic to s. On the space s, consider the operator A from Example 2.8, and let $\tilde{A} = F^{-1}AF \oplus 0$, where F is an isomorphism X_1 onto s. Then the operator $\tilde{A} \in \mathcal{L}(X, X)$ is weakly locally nilpotent but it is not nilpotent.

Sufficiency. Let X not contain a closed subspace isomorphic to s. Then, by the Bessaga–Pelczyński theorem [4, Theorem 2], there exists a continuous norm $\|\cdot\|$ on X. Consider the normed space $(X, \|\cdot\|)$ and its dual space $(X, \|\cdot\|)^*$ which is a Banach space. It is obvious that the space $(X, \|\cdot\|)^*$ is contained in X^*. Now, let $A \in \mathcal{L}(X, X)$ be a weakly locally nilpotent operator on the space X. Then A is a weakly locally nilpotent operator on the space $(X, \|\cdot\|)$, too. By Lemma 2.6, the operator A is a locally nilpotent operator on the space $(X, \|\cdot\|)$ and thus on the original space X. Since $A \in \mathcal{L}(X, X)$, we have that A is nilpotent (see Remark 2.3). The proof is complete. □

3. Conditions for the existence and uniqueness of a solution of implicit difference equations in Banach and Fréchet spaces

We start with proving the following general fact on the existence and uniqueness for the solution of (1.2).

Suppose that Z is a locally convex Hausdorff space. We say that Banach's inverse mapping theorem is valid for the space Z if the operator $\mathcal{T}^{-1}$ is continuous for any continuous isomorphism $\mathcal{T}$ of the space Z.

Lemma 3.1. *Let X be a locally convex Hausdorff space. Let Banach's inverse mapping theorem be valid for $S(X)$. If the difference equation (1.2) has a unique solution for any sequence $\{f_n\}_{n=0}^{\infty}$, then the operator $A : X \to X$ is weakly locally nilpotent. In this case, the solution of (1.2) is given by (1.3), where the series in the right-hand side of (1.3) converges in the topology of X. Moreover, this solution of (1.2) continuously depends on the right-hand side of this equation in the topology of $S(X)$.*

Proof. Since $A \in \mathcal{L}(X, X)$, we have that the linear operator

$$\mathcal{A} : S(X) \to S(X), \quad (\mathcal{A}x)_n = x_n - Ax_{n+1}, \quad x \in S(X),$$

is continuous on the space $S(X)$. Now, by the assumption of the lemma, the operator $\mathcal{A}^{-1} : S(X) \to S(X)$ is continuous. In particular, the solution of equation (1.2) continuously depends on the right-hand side of this equation in the topology of $S(X)$.

For $m \in \mathbb{N}$ and $f = \{f_n\}_{n=0}^{\infty} \in S(X)$, we consider the sequence $f^m = \{f_n^m\}_{n=0}^{\infty} \in S(X)$, where

$$f_n^m = \begin{cases} f_n, & n \le m \\ 0, & n > m \end{cases}.$$

By the immediate verification, we conclude that for $m \in \mathbb{N}$ the sequence $x^m = \{x_n^m\}_{n=0}^{\infty}$, where

$$x_n^m = \begin{cases} \sum\limits_{k=0}^{m-n} A^k f_{n+k}, & n \le m \\ 0, & n > m, \end{cases} \tag{3.1}$$

is a solution of the difference equation

$$Ax_{n+1}^m = x_n^m - f_n^m, \quad n = 0, 1, 2, \dots. \tag{3.2}$$

Note that $f_{n+k}^m = f_{n+k}$ if $k \le m - n$ and $m \ge n$. Formula (3.2) shows that $\mathcal{A}^{-1} f^m = x^m$. Since the operator $\mathcal{A}^{-1}$ is continuous and $\lim_{m\to\infty} f^m = f$ in $S(X)$, then there exists $\lim_{m\to\infty} x^m$ in the space $S(X)$. In particular, this implies that there exists $\lim_{m\to\infty} x_0^m$. Taking into account the representation (3.1), for x_0^m we obtain that the series $\sum_{k=0}^{\infty} A^k f_k$ converges for any $f_k \in X$ $(k = 0, 1, 2, \dots)$ to some element x_0. In a similar way, the elements x_n $(n = 1, 2, \dots)$ are defined by formula (1.3). By the direct substitution of these elements in equation (1.2), we verify that formula (1.3) defines the solution of equation (1.2).

Now we prove the weak local nilpotency of A. Put $f_k = \alpha_k f$, where f is an arbitrary element of X and $\{\alpha_k\}_{k=0}^{\infty}$ is an arbitrary sequence of real numbers. Then, for any linear continuous functional $\varphi \in X^*$, the scalar series $\sum_{k=0}^{\infty} \alpha_k \varphi(A^k f)$ converges. Consequently, $\lim_{k\to\infty} \alpha_k \varphi(A^k f) = 0$. Since $\{\alpha_k\}_{k=0}^{\infty}$ is an arbitrary sequence, we obtain that for any $f \in X$, $\varphi \in X^*$ there exists $k = k(f, \varphi) \in \mathbb{N}$ such that $\varphi(A^k f) = 0$. Therefore, A is a weakly locally nilpotent operator. The lemma is proved. $\square$

Example 3.2. Suppose the difference equation (1.2) has a unique solution for any sequence $\{f_n\}_{n=0}^{\infty}$. Let us prove that the operator $I - A$ is invertible. If $(I - A)u = 0$, then $Au = u$ and the assumption on the uniqueness for a solution of (1.2) implies $u = 0$. Now, let $g \in X$. Consider (1.2) with $f_n = g$ for all $n = 0, 1, 2, \ldots$. If a sequence $\{x_n\}_{n=0}^{\infty}$ is a solution of this equation, then so is $\{x_{n+1}\}_{n=0}^{\infty}$. Therefore, $x_{n+1} = x_n$ for all n, i.e., $\{x_n\}_{n=0}^{\infty}$ is a constant sequence, and $(I - A)x_0 = g$. Thus we have that the operator $I - A$ is invertible and the unique solution of the equation $Ax_{n+1} = x_n - f_0$ has the form

$$x_n = (I - A)^{-1} f_0, \quad n = 0, 1, 2, \ldots.$$

Now, if the all conditions of Lemma 3.1 are fulfilled, then this Lemma implies that

$$(I - A)^{-1} f_0 = \sum_{k=0}^{\infty} A^k f_0.$$

Remark 3.3. It became clear while proving Lemma 3.1 that the continuous operator A in question has the following property: the series $\sum_{k=0}^{\infty} A^k f_k$ converges for any sequence $\{f_k\}_{k=0}^{\infty}$. This property can be also treated as a certain generalization of the nilpotency property. In fact, the proof of Lemma 3.1 establishes that any operator with such property is weakly locally nilpotent.

Remark 3.4. It has been shown in Introduction that the general form (1.3) for a solution of (1.2) can be deduced via the general concepts of linear analysis. It should be also observed that it is derivable by using an analog of the Cramer formula for a solution of linear systems. Let us view the elements of the space $S(X)$ as column vectors. Rewrite (1.2) in the form

$$\mathcal{A}x = f, \quad \text{with } \mathcal{A} = \begin{pmatrix} I & -A & 0 & 0 & \cdots \\ 0 & I & -A & 0 & \cdots \\ 0 & 0 & I & -A & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad x, f \in S(X). \tag{3.3}$$

We claim that the form (1.3) for a solution of (1.2) can be considered as the collection of Cramer's formulas for a solution of the infinite system of linear equations (3.3). In this context, it is natural to assume that the determinant Δ of this system is the identity operator I. Consider the following operator-vector matrix $\mathcal{A}_0$ corresponding to the vector x_0 which is to be found:

$$\mathcal{A}_0 = \begin{pmatrix} f_0 & -A & 0 & 0 & \cdots \\ f_1 & I & -A & 0 & \cdots \\ f_2 & 0 & I & -A & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Introduce now the sequence of principal minors of this matrix in the sense of [9]:

$$\Delta_{0,0} = f_0,$$
$$\Delta_{0,1} = \begin{vmatrix} f_0 & -A \\ f_1 & I \end{vmatrix} = f_0 + Af_1,$$
$$\Delta_{0,2} = \begin{vmatrix} f_0 & -A & 0 \\ f_1 & I & -A \\ f_2 & 0 & I \end{vmatrix} = f_0 + Af_1 + A^2 f_2,$$
$$\dots$$
$$\Delta_{0,m} = \begin{vmatrix} f_0 & -A & 0 & 0 & \cdots & 0 \\ f_1 & I & -A & 0 & \cdots & 0 \\ f_2 & 0 & I & -A & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ f_m & 0 & 0 & 0 & \cdots & I \end{vmatrix} = \sum_{k=0}^{m} A^k f_k.$$

Under the assumptions of Lemma 3.1 and in view of (1.3), there exists $\lim_{m\to\infty} \Delta_{0,m}$, which we call *the determinant of* $\mathcal{A}_0$ and denote by $\Delta_0(f)$. Now (1.3) for the vector x_0 can be written as the Cramer formula

$$x_0 = \Delta^{-1} \cdot \Delta_0(f).$$

Similar techniques are applicable for finding other components of the solution. In view of (1.3), the operator inverse to $\mathcal{A}$ is given by

$$\mathcal{A}^{-1} = \begin{pmatrix} I & A & A^2 & A^3 & \cdots \\ 0 & I & A & A^2 & \cdots \\ 0 & 0 & I & A & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Now assume that X is a Fréchet space. The following theorem establishes the necessary and sufficient conditions for a unique solvability of (1.2) with an arbitrary sequence $\{f_n\}_{n=0}^{\infty}$.

Theorem 3.5. *Let X be a Fréchet space and $A \in \mathcal{L}(X, X)$. The local nilpotency of the adjoint operator A^* is a necessary condition for the existence and uniqueness of solution of the difference equation (1.2) with any sequence $\{f_n\}_{n=0}^{\infty}$. Under the additional assumption that X is weakly sequentially complete, this condition is also sufficient. In this case, the solution of (1.2) is given by (1.3), where the convergence of series in the right-hand side of (1.3) is in the topology of the Fréchet space X.*

Proof. Necessity. The space $S(X)$ is a Fréchet space with the distance [15, p. 28],

$$d\big(\{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty}\big) = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{d_X(x_n, y_n)}{1 + d_X(x_n, y_n)}.$$

In this space the Banach inverse mapping theorem is valid [6, Chapter 1, §3, Section 3]. Then, by Lemma 3.1, the operator A is weakly locally nilpotent. Since X is a Fréchet space, we obtain by Lemma 2.5 that A^* is a locally nilpotent operator.

Sufficiency. We consider the vector-valued sequence

$$y_m = \sum_{k=0}^{m} A^k f_k \in X.$$

By the local nilpotency of A^*, the scalar sequence

$$\varphi(y_m) = \sum_{k=0}^{m} \varphi(A^k f_k) = \sum_{k=0}^{m} (A^*)^k \varphi(f_k)$$

is stabilized for any functional $\varphi \in X^*$: $\varphi(y_m) = \varphi(y_{k(\varphi)})$, $m \geq k(\varphi)$. Therefore the sequence $\{y_m\}_{m=0}^{\infty}$ is fundamental with respect to the weak convergence on X. Since the space X is weakly sequentially complete, then there exists an element $x_0 \in X$ such that the sequence $\{y_m\}_{m=0}^{\infty}$ weakly converges to x_0. Then the series $\sum_{k=0}^{\infty} A^k f_k$ weakly converges to the element x_0. Analogously, for $n = 1, 2, \ldots$ the sequence $\{x_n\}_{n=0}^{\infty}$ is defined by (1.3). Taking into account the continuity of A, by a direct substitution of (1.3) into (1.2), we verify that the sequence $\{x_n\}_{n=0}^{\infty}$ is a solution of (1.2).

We prove the uniqueness of a solution of equation (1.2). For this, we consider the homogeneous equation

$$Ax_{n+1} = x_n, \quad n = 0, 1, 2, \ldots. \tag{3.4}$$

Then $x_n = A^k x_{n+k}$, $\quad n, k = 0, 1, 2, \ldots$. Taking into account the local nilpotency of A^*, we obtain $\varphi(x_n) = 0$ $(n = 0, 1, 2, \ldots)$ for any functional $\varphi \in X^*$. By the Hahn–Banach theorem, $x_n = 0$, $n = 0, 1, 2, \ldots$, i.e., the homogeneous equation (3.4) has only trivial solution. Therefore the difference equation (1.2) has a unique solution for any sequence $\{f_n\}_{n=0}^{\infty}$. It follows from the proven necessity of the theorem assertion that the series $\sum_{k=0}^{\infty} A^k f_k$ converges in the topology of the space X for any $f_k \in X$ $(k = 0, 1, 2, \ldots)$. Consequently, for any $n \in \mathbb{N}$, the series in the right-hand side of equality (1.3) also converges in the topology of the space X. The proof is complete. □

Remark 3.6. Example 2.8 of the shift operator on the Fréchet space $X = s$ shows that (1.2) may have a unique solution for an arbitrary sequence $\{f_n\}_{n=0}^{\infty}$, while the operator A is not locally nilpotent. In fact, with the elements from s being written as $x_n = \{x_{n,m}\}_{m=0}^{\infty}$, $f_n = \{f_{n,m}\}_{m=0}^{\infty}$, (1.2) acquires the form

$$x_{n,0} = f_{n,0}, \quad x_{n+1,m-1} = x_{n,m} - f_{n,m}, \quad m \in \mathbb{N}, \ n = 0, 1, 2, \ldots. \tag{3.5}$$

Now, for an arbitrary $f_n = \{f_{n,m}\}_{m=0}^{\infty} \in s$ $(n = 0, 1, 2, \ldots)$, (1.2) has a unique solution $x_n = \{x_{n,m}\}_{m=0}^{\infty} \in s$ $(n = 0, 1, 2, \ldots)$, which is deducible from the relations (3.5),

$$x_{n,m} = \sum_{k=0}^{m} f_{n+k,m-k}, \quad n, m = 0, 1, 2, \ldots.$$

The following assertion claims that in the particular case, when X is a Banach space, the existence and uniqueness of a solution of (1.2) for any preassigned sequence $\{f_n\}_{n=0}^{\infty}$ implies the nilpotency of the operator A.

Corollary 3.7. *Let X be a Banach space and $A \in \mathcal{L}(X, X)$. The difference equation (1.2) has a unique solution $\{x_n\}_{n=0}^{\infty}$ for an arbitrary sequence $\{f_n\}_{n=0}^{\infty}$ iff A is nilpotent. Moreover, with $r + 1$ being the nilpotency index of A, the solution of the difference equation (1.2) is given by (2.1).*

Proof. Sufficiency follows from Lemma 2.1. We prove the *necessity*. By Theorem 3.5, the series $\sum_{k=0}^{\infty} A^k f_k$ converges for any $f_k \in X$ $(k = 0, 1, 2, \dots)$. We show that it is possible only in the case of nilpotency of A. Actually, if there exists $x \in X$ such that $A^k x \neq 0$ $(k = 1, 2, 3, \dots)$, then the series $\sum_{k=0}^{\infty} A^k f_k$ diverges for $f_k = \frac{x}{\|A^k x\|}$. Therefore, for any $x \in X$, there exists $k = k(x) \in \mathbb{N}$ such that $A^k x = 0$, i.e., A is locally nilpotent. By Remark 2.3, the operator A is nilpotent. The corollary is proved. □

Corollary 3.8. *Let X be a complex Hilbert space, and $A \in \mathcal{L}(X, X)$ be a normal operator, i.e., $A^*A = AA^*$. Then the difference equation (1.2) has a unique solution $\{x_n\}_{n=0}^{\infty}$ for an arbitrary sequence $\{f_n\}_{n=0}^{\infty}$ iff $A = 0$.*

To prove this fact, it suffices to note that any nilpotent normal operator is the null-operator.

Corollary 3.7 admits the following generalization.

Corollary 3.9. *Let X be a Fréchet space where there is a norm which is continuous with respect to the topology of the space X. Let $A \in \mathcal{L}(X, X)$. The difference equation (1.2) has a unique solution for every sequence $\{f_n\}_{n=0}^{\infty}$ iff the operator A is nilpotent.*

To prove this fact, we note that by Theorem 3.5 and Lemma 2.5, the operator A is weakly locally nilpotent. Therefore the nilpotency of A follows from Theorem 2.9. The converse is just the claim of Lemma 2.1.

Remark 3.10. It is interesting to observe that if a Fréchet space admits no continuous norm, then, by the Bessaga–Pelczyński theorem, it must contain a subspace isomorphic to the space of all sequences (see [4, Theorem 2] and [14, Exercize 7, §16.3.4]).

Now let us turn to studying the general equation (1.1).

Theorem 3.11. *Let X and Y be Fréchet spaces and $A, B \in \mathcal{L}(X, Y)$. The existence of the inverse operator $B^{-1} \in \mathcal{L}(Y, X)$ and the local nilpotency of $(B^{-1}A)^*$ is a necessary condition for the existence and uniqueness of solution of the difference equation (1.1) for an arbitrary sequence $\{g_n\}_{n=0}^{\infty}$. Under the additional assumption of X being weakly sequentially complete, this condition also appears to be sufficient. In this case, a solution of (1.1) is given by*

$$x_n = \sum_{k=0}^{\infty} (-1)^k (B^{-1}A)^k B^{-1} g_{n+k}, \quad n = 0, 1, 2, \dots, \tag{3.6}$$

where the convergence of series in the right-hand side of (3.6) *is in the topology of the Fréchet space* X*.*

Proof. Necessity. Let the right-hand side of equation (1.1) possess the property $g_n = 0,\ n = 1, 2, \dots$. Then the corresponding unique solution $\{x_n\}_{n=1}^{\infty}$ of this equation satisfies to the homogeneous equation

$$Ax_{n+1} + Bx_n = 0, \quad n = 1, 2, \dots.$$

By the uniqueness of a solution for equation (1.1), we have $x_n = 0, \quad n = 1, 2, \dots$. Then the vector x_0 is a solution of the linear equation $Bx_0 = g_0$. This equation has a unique solution for any $g_0 \in Y$. Then, by the Banach inverse mapping theorem, there exists an inverse operator $B^{-1} \in \mathcal{L}(Y, X)$. Therefore the difference equation (1.1) is reduced to the equation

$$-B^{-1}Ax_{n+1} = x_n - B^{-1}g_n, \quad n = 0, 1, 2, \dots. \tag{3.7}$$

Applying Theorem 3.5 to (3.7), we obtain the local nilpotency of the operator $(B^{-1}A)^*$ and representation (3.6) for the corresponding solution.

Sufficiency. It follows from the existence of the inverse operator $B^{-1} \in \mathcal{L}(Y, X)$ that equation (1.1) is equivalent to equation (3.7). It follows from the local nilpotency of the operator $(B^{-1}A)^*$ and Theorem 3.5 that this equation has a unique solution for any sequence $\{g_n\}_{n=0}^{\infty}$ and this solution is represented by (3.6). The theorem is proved. □

Corollary 3.12. *Under assumptions of Theorem* 3.11*, the initial problem* $x_0 = a \in X$ *for* (1.1) *is solvable iff*

$$a = \sum_{k=0}^{\infty} (-1)^k (B^{-1}A)^k B^{-1} g_k.$$

In the case where X, Y are Banach spaces, we obtain the assertion below, which is similar to Corollary 3.7.

Corollary 3.13. *Let* X *and* Y *be Banach spaces and* $A, B \in \mathcal{L}(X, Y)$*. The difference equation* (1.1) *has a unique solution* $\{x_n\}_{n=0}^{\infty}$ *for an arbitrary sequence* $\{g_n\}_{n=0}^{\infty}$ *iff there exists the inverse operator* $B^{-1} \in \mathcal{L}(Y, X)$ *and the operator* $B^{-1}A$ *is nilpotent. In this case, if the nilpotency index of* $B^{-1}A$ *is equal to* $r + 1$*, then the solution of the difference equation* (1.1) *is determined by*

$$x_n = \sum_{k=0}^{r} (-1)^k (B^{-1}A)^k B^{-1} g_{n+k}, \quad n = 0, 1, 2, \dots.$$

The proof is similar to that of Theorem 3.11 with a reference to Corollary 3.7.

4. An implicit difference equation in the dual space

It was noted in Remark 3.6 that (1.2) may have a unique solution for any sequence $\{f_n\}_{n=0}^{\infty} \in S(X)$, while the operator A is not locally nilpotent. However, one has an important case when the property of local nilpotency is valid.

Let V be a locally convex Hausdorff space and $T \in \mathcal{L}(V,V)$. Suppose that the dual space V^* is endowed with the *strong topology*, i.e., the topology of uniform convergence on every bounded subset of V [6, Chapter IV, §3, Section 1]. According to [6, Chapter IV, §4, Section 2], the adjoint operator $T^* : V^* \to V^*$ is continuous. Consider the implicit difference equation in the dual space V^*:

$$T^*\varphi_{n+1} = \varphi_n - \psi_n, \quad n = 0, 1, 2, \dots \tag{4.1}$$

with a given sequence $\{\psi_n\}_{n=0}^{\infty} \in S(V^*)$. Theorem 3.5 implies the following criterion of the existence and uniqueness for a solution of (4.1).

Theorem 4.1. *Let V be a locally convex Hausdorff space. Assume that the strong dual V^* is a Fréchet space and $T \in \mathcal{L}(V,V)$. The local nilpotency for T is a necessary condition for the existence and uniqueness of solution of the difference equation (4.1) with an arbitrary sequence $\{\psi_n\}_{n=0}^{\infty}$; if V is a reflexive space, this condition is also sufficient.*

Proof. Equation (4.1) is a particular case of equation (1.2) with the operator $A = T^*$ acting in the Fréchet space $X = V^*$. Here $X^* = V^{**}$ and $A^* = T^{**}$.

We prove an assertion on the *necessity* of the condition of the theorem. Let equation (4.1) have a unique solution for any sequence $\{\psi_n\}_{n=0}^{\infty}$. Then, by Theorem 3.5, the operator A^* is locally nilpotent. For any $x \in V$, we define the linear continuous functional $\varphi_x \in V^{**}$ as follows: $\varphi_x(f) = f(x)$, $f \in V^*$. By the local nilpotency of the operator $A^* = T^{**}$, for any $x \in X$, there exists a number $k = k(x)$ such that for any $f \in V^*$ the following equalities hold:

$$0 = (T^{**})^k \varphi_x(f) = \varphi_x((T^*)^k f) = (T^*)^k f(x) = f(T^k x).$$

It follows from the Hahn–Banach theorem that $T^k x = 0$. Thus the operator T is locally nilpotent.

Sufficiency. Let T be a locally nilpotent operator and V be a reflexive space. Then V is a barrelled space [6, Chapter IV, §3, Section 3]. We show that V^* is a weakly sequentially complete space. Consider an arbitrary element $v \in V$ and a sequence $\{x_n\}_{n=1}^{\infty}$ of elements from $X = V^*$ such that the sequence $\{x_n(v)\}_{n=1}^{\infty}$ is fundamental. Since V is a barrelled space, it follows from Corollary 7.1.4 [8, Section 7.1] that the linear functional $x(v) = \lim_{n\to\infty} x_n(v)$ is continuous. Now it follows from the reflexivity of V that V^* is weakly sequentially complete. Furthermore, since the space V is reflexive, then the operator $A^* = T^{**}$ can be identified with the original operator T [8, Section 8.7]. By Theorem 3.5, there exists a unique solution of equation (4.1) for any sequence $\{\psi_n\}_{n=0}^{\infty}$. The theorem is proved. $\square$

Example 4.2. Consider the locally convex Hausdorff vector space $V = \mathbb{R}^\infty$ of finite sequences of real numbers with the natural inductive limit topology of finite-dimensional subspaces. Note that any linear functional on $\mathbb{R}^\infty$ is continuous and the strong dual space V^* can be identified with the Fréchet space s. The space V^{**} is naturally identified with the original space V (see [15, p. 119]). Therefore V is a reflexive space. Let $T : \mathbb{R}^\infty \to \mathbb{R}^\infty$ be a linear operator defined everywhere. T is certainly continuous. Consider the difference equation (4.1) in the space s with a given sequence $\{\psi_n\}_{n=0}^\infty \in S(V^*)$. By Theorem 4.1, equation (4.1) has a unique solution for an arbitrary sequence $\{\psi_n\}_{n=0}^\infty \in S(V^*)$ iff T is locally nilpotent.

5. A necessary condition for solvability of an implicit difference equation in an (LF)-space

Let X be an (LF)-space, i.e., a strict inductive limit of a sequence of Fréchet spaces [21, Chapter II, Section 6]. The following theorem establishes a necessary condition for unique solvability of (1.2) for any preassigned sequence $\{f_n\}_{n=0}^\infty$.

Theorem 5.1. *Let X be an (LF)-space. If the difference equation (1.2) has a unique solution for an arbitrary sequence $\{f_n\}_{n=0}^\infty$, then the operator $A \in \mathcal{L}(X, X)$ is weakly locally nilpotent. In this case, the solution of (1.2) is given by (1.3), where the convergence of series in the right-hand side of (1.3) is in the original topology of X.*

Proof. First, let us show that X is an ultrabornological space [18, Definition 13.2.3 (b)]. It should be noted that a Fréchet space is ultrabornological [18, Example 13.2.8(d)]. Furthermore any inductive limit of ultrabornological spaces is again ultrabornological [18, Theorem 13.2.9]. Hence X is ultrabornological. Note also that the space X is sequentially complete. Since the topological product of countably many bornological spaces is again bornological [15, p. 384] and the product of countably many sequentially complete spaces is again sequentially complete [15, p. 296], we have that $S(X)$ is an ultrabornological space [18, Theorem 13.2.12]. Consequently, the space $S(X)$ is represented as an inductive limit of Banach spaces (see [18, Theorem 13.2.11]). Next, the space $S(X)$ is a countable topological degree of X and X is covered by countably many its Fréchet subspaces. By the terminology of [19, p. 231], this means that $S(X)$ is a $\mathcal{PUF}$-space. Therefore, by the Banach open mapping theorem, for $\mathcal{PUF}$-spaces [19, p. 231, Theorem 3] we obtain that the Banach open mapping theorem is fulfilled for the space $S(X)$. By Lemma 3.1, the operator A is weakly locally nilpotent. The theorem is proved. □

Corollary 5.2. *Under assumptions of Theorem 5.1, suppose that the strong dual X^* is a Fréchet space. If the difference equation (1.2) has a unique solution for an arbitrary sequence $\{f_n\}_{n=0}^\infty$, then the operator $A \in \mathcal{L}(X, X)$ is locally nilpotent. In this case, the solution of (1.2) is given by (1.3), where the convergence of series in the right-hand side of (1.3) is in the original topology of X.*

This is an immediate consequence of Theorem 5.1 and Lemma 2.6.

Example 5.3. Consider the space $X = \mathbb{R}^\infty$ of finite sequences with the natural topology of the inductive limit of finite-dimensional subspaces. The dual space X^* for X is the Fréchet space s. Let $A : \mathbb{R}^\infty \to \mathbb{R}^\infty$ be an arbitrary linear operator defined everywhere. A is certainly continuous. Consider (1.2) in $\mathbb{R}^\infty$ with a preassigned sequence $\{f_n\}_{n=0}^\infty \in S(X)$. By Corollary 5.2, the difference equation (1.2) may have a unique solution for an arbitrary sequence $\{f_n\}_{n=0}^\infty \in S(X)$ in the case of the local nilpotency of A only.

Remark 5.4. The subject of Example 5.3 is the difference equation (1.2) in the space $\mathbb{R}^\infty$, related to the context of Theorem 5.1 and its corollary. What is crucial here is the possibility to apply the Banach inverse mapping theorem to $S(\mathbb{R}^\infty)$. It follows from Köthe results (see [16, p. 31]) that $S(\mathbb{R}^\infty)$ is a barrelled Pták space, and hence the Banach inverse mapping theorem is applicable to $S(\mathbb{R}^\infty)$ (see Section 6 below). It is interesting to observe that certain sophisticated results of the topological vector spaces theory, in fact, work in the *purely algebraic context* of Example 5.3.

The following example shows that, in general, a local nilpotency of the operator A is not sufficient for the uniqueness of a solution of (1.2).

Example 5.5. Let $X = \mathbb{R}^\infty$, and A be the left shift operator, i.e., $Au = (u_1, u_2, \dots)$ for $u = (u_0, u_1, u_2, \dots) \in X$. Now (1.2) acquires the form

$$x_{n+1,m+1} = x_{n,m} - f_{n,m}, \quad n, m = 0, 1, 2, \dots, \tag{5.1}$$

where $x_n = \{x_{n,m}\}_{m=0}^\infty$ and $f_n = \{f_{n,m}\}_{m=0}^\infty$ are the elements of $\mathbb{R}^\infty$. The difference equation (5.1) is equivalent to

$$x_{n+1,m+1} = \begin{cases} x_{n-m,0} - \sum\limits_{k=0}^{m} f_{n-k,m-k}, & m \le n \\ x_{0,m-n} - \sum\limits_{k=0}^{n} f_{n-k,m-k}, & m > n \end{cases}. \tag{5.2}$$

Choose an arbitrary sequence $\{x_{0,m}\}_{m=0}^\infty \in \mathbb{R}^\infty$ and an arbitrary sequence $\{x_{n,0}\}_{n=1}^\infty$. Now all other $x_{n,m}$'s are uniquely determined via (5.2). Then, for each $n \in \mathbb{N}$, the sequence $\{x_{n,m}\}_{m=0}^\infty$ is an element of $\mathbb{R}^\infty$, too. In fact, for any fixed $n = 0, 1, 2, \dots$ there exists $m_0(n) > n$ such that $f_{n,m} = 0$ for all $m > m_0(n)$. Since $\{x_{0,m}\}_{m=0}^\infty \in \mathbb{R}^\infty$, we deduce the existence of $k_0 \in \mathbb{N}$ such that $x_{0,m} = 0$ for all $m > k_0$. Set

$$N_0(n) = \max\{\max_{k=0,\dots,n}(m_0(n-k) + k), n + k_0\}.$$

It follows from (5.2) that $x_{n+1,m+1} = 0$ for all $m > N_0(n)$. Therefore, $\{x_{n,m}\}_{m=0}^\infty \in \mathbb{R}^\infty$ for any $n \in \mathbb{N}$.

The elements $x_{n,m}$ constructed above satisfy the difference equation (5.1). Therefore, given arbitrary $\{f_n\}_{n=0}^\infty \in S(\mathbb{R}^\infty)$ and initial vector $x_0 \in \mathbb{R}^\infty$, the initial problem for (1.2) has a solution. In particular, the associated homogeneous equation has nontrivial solutions. Among those, one has $x_{n,m} = \delta_{nm}$, where δ_{nm} is the Kronecker delta.

6. A necessary condition for solvability of an implicit difference equation in a Pták space

This section expounds an application of the theory of Pták spaces intended to deduce necessary conditions for the existence and uniqueness of a solution of the difference equation (1.2) for an arbitrary sequence $\{f_n\}_{n=0}^{\infty}$.

We recall the definition of a Pták space or a **B**-complete space (see, for example, [21, Chapter IV, Section 8], [16, §34]). Let X be a locally convex Hausdorff space. We denote by $\sigma(X^*, X)$ the weak-star topology on the dual space X^*. The space X is said to be a *Pták space* if a subspace $Q \subset X^*$ is closed for $\sigma(X^*, X)$ whenever $Q \cap F$ is $\sigma(X^*, X)$-closed in F for each equicontinuous set $F \subset X^*$ (see [21, Chapter III, Section 4]). We note that every Fréchet space is a Pták space [21, Chapter IV, Section 8, Example 1] and every closed subspace of a Pták space is a Pták space [21, Chapter IV, 8.2].

The following theorem establishes a necessary condition for unique solvability of equation (1.2) in a Pták space with an arbitrary preassigned sequence $\{f_n\}_{n=0}^{\infty}$.

Theorem 6.1. *Let X be such a barrelled space that its countable topological power is a Pták space. If the difference equation (1.2) has a unique solution for an arbitrary sequence $\{f_n\}_{n=0}^{\infty}$, then A is a weakly locally nilpotent operator. Moreover, the solution of (1.2) is given by (1.3), where the convergence of series in the right-hand side of (1.3) is in the original topology of X.*

Proof. The space $S(X)$ is a countable topological degree of the space X. According to [6, Chapter 4, §2, Section 2, Remark 1], $S(X)$ is a barrelled space. Therefore $S(X)$ is a barreled Pták space. Now, by the open mapping theorem for barreled Pták spaces [21, Chapter IV, Section 8, Corollary 1], we obtain that the Banach inverse mapping theorem is fulfilled in the space $S(X)$. By Lemma 3.1, the operator A is weakly locally nilpotent. The theorem is proved. □

Example 6.2. With M being an infinite set, consider the space $X = \mathbb{R}^M$, which is a topological power of a one-dimensional space. X is a locally convex barreled space. Observe that with M being uncountable, X fails to be a Fréchet space [15, p. 207]. It should be noted that $S(X)$ is isomorphic to $\mathbb{R}^{M\times(\mathbb{N}\cup\{0\})}$. Now [21, Chapter IV, Section 8, Example 3] implies that $S(X)$ is a Pták space. Therefore, if A is an arbitrary continuous linear operator in X, then the existence and the uniqueness for a solution of (1.2) with an arbitrary sequence $\{f_n\}_{n=0}^{\infty} \in S(X)$ can be true in the case of the weak local nilpotency of A only.

Corollary 6.3. *Let X be a weakly complete locally convex space and $A \in \mathcal{L}(X, X)$. If the difference equation (1.2) has a unique solution with an arbitrary sequence $\{f_n\}_{n=0}^{\infty}$, then A is weakly locally nilpotent. Moreover, the solution of (1.2) is given by (1.3), where the convergence of series in the right-hand side of (1.3) is in the original topology of X.*

Proof. According to [21, Exercise 6a to Chapter IV], the space X is topologically isomorphic to the space $\mathbb{R}^M$ for some set M. Now the assertion of Corollary 6.3 follows from the arguments given in Example 6.2 and Theorem 6.1. □

Remark 6.4. The assumption of Theorem 6.1 was that $S(X)$ is a Pták space. It was shown in [16, p. 31] that the product of Pták spaces may appear not to be a Pták space. However, with $S(X)$ being a Pták space, one deduces from [21, Chapter IV, Section 8, Theorem 8.2, Corollary 2] that X is a Pták space too.

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Неявне лінійне неоднорідне різницеве рівняння у банахових та локально опуклих просторах

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Темою дослідження цієї роботи є неявні лінійні різницеві рівняння $Ax_{n+1} + Bx_n = g_n$ та $Ax_{n+1} = x_n - f_n$, $n = 0, 1, 2, \ldots$, де A та B є неперервними операторами, які діють на деяких локально опуклих просторах. Одержано необхідні та достатні умови разом з явними формулами для розв'язків цих рівнянь. Як застосування загальної теорії, вивчено рівняння $Ax_{n+1} = x_n - f_n$ у просторі $\mathbb{R}^\infty$ фінітних послідовностей та у просторі $\mathbb{R}^M$, де M — довільна множина.

Ключові слова: різницеве рівняння, локально опуклий простір, банахів простір, локально нільпотентний оператор.