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# Fluctuations of the Process of Moduli for the Ginibre and Hyperbolic Ensembles

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*To Vladimir Alexandrovich Marchenko, with admiration and love.*

We investigate the point process of moduli of the Ginibre and hyperbolic ensembles. We show that far from the origin and at an appropriate scale, these processes exhibit Gaussian and Poisson fluctuations. Among the possible Gaussian fluctuations, we can find white noise but also fluctuations with non-trivial covariance at a particular scale.

*Key words:* Ginibre ensemble, hyperbolic ensemble, process of moduli, normality, white noise, Poisson point process

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## 1. Formulation of the main results

The main results of this paper, Theorems 1.3–1.8, establish limit theorems for additive statistics of the Ginibre ensemble and the hyperbolic ensembles, introduced by Krishnapur, including the determinantal point process with the Bergman kernel, which, by the Peres–Virág theorem, is the zero set of the Gaussian analytic function on the unit disc.

In this section, we begin by recalling the notion of determinantal point process, which are point processes where the correlation functions take the form of a determinant. Afterwards, the specific examples we are interested in, namely the Ginibre point process and the hyperbolic ensembles, are discussed. Finally, we state the main results of this note, Theorems 1.3–1.8.

**1.1. Determinantal point process.** Let $X$ be a locally compact Polish space and $\mathcal{B}_0(X)$ the collection of all pre-compact Borel subsets of $X$. We shall denote by $\mathrm{Conf}(X)$, the space of all locally finite configurations over $X$, that is,

$$\mathrm{Conf}(X) := \left\{\xi = \textstyle\sum_i \delta_{x_i} : \forall i, x_i \in X \text{ and } \xi(\Delta) < \infty \text{ for all } \Delta \in \mathcal{B}_0(X)\right\}.$$

We shall consider this set endowed with the vague topology, i.e., the weakest topology on $\mathrm{Conf}(X)$ such that for any compactly supported continuous function $f$ on $X$, the map $\mathrm{Conf}(X) \ni \xi \mapsto \int_X f \mathrm{d}\xi$ is continuous. It can be seen that the configuration space $\mathrm{Conf}(X)$ equipped with the vague topology turns out to be a Polish space. Additionally, it can be seen that the Borel $\sigma$-algebra $\mathcal{F}$ on $\mathrm{Conf}(X)$ is generated by the cylinder sets $C_n^{\Delta} = \{\xi \in \mathrm{Conf}(X) : \xi(\Delta) = n\}$, where $n \in \mathbb{N} = \{0, 1, 2, \dots\}$ and $\Delta \in \mathcal{B}_0(X)$. Finally, a point process on $X$ will be a measurable map

$$\mathscr{X} : (\Omega, \mathcal{F}(\Omega), \mathbb{P}) \to (\mathrm{Conf}(X), \mathcal{F}),$$

where $(\Omega, \mathcal{F}(\Omega), \mathbb{P})$ is any probability space. For further background, see [5,10,13].

A point process $\mathscr{X}$ is called simple if it almost surely assigns at most measure one to singletons. In the simple case, $\mathscr{X}$ can be identified with a random discrete subset of $X$ and for any Borel set $\Delta$ on $X$, the number $\mathscr{X}(\Delta) \in \mathbb{N} \cup \{\infty\}$ represents the number of points of this discrete subset that fall in $\Delta$. So, for instance, we will use the notation $\{T(x) : x \in \mathscr{X}\}$ instead of the usual pushforward notation $T_*\mathscr{X}$ for simplicity.

Determinantal point processes have been introduced by Odile Macchi [14] in the seventies. We recall the definition. Let $\mu$ be a Radon measure on $X$ and let $K : X \times X \to \mathbb{C}$ be a measurable function. A simple point process $\mathscr{X}$ is called determinantal on $X$ associated to the kernel $K$ with respect to the reference measure $\mu$ if, for every $k \in \mathbb{N}_+$ and any family of mutually disjoint subsets $\Delta_1, \Delta_2, \dots, \Delta_k \in \mathcal{B}_0(X)$,

$$\mathbb{E}\left[\prod_{i=1}^{k} \mathscr{X}(\Delta_i)\right] = \int_{\Delta_1 \times \cdots \times \Delta_k} \det\left[K(x_i, x_j)\right]_{1 \le i,j \le k} \mathrm{d}\mu(x_1) \cdots \mathrm{d}\mu(x_k). \tag{1.1}$$

See, e.g., [1–4, 9, 15–18] for further background of determinantal point process.

The moments of the linear statistics $\sum_{x \in \mathscr{X}} f(x) := \int_X f \,\mathrm{d}\mathscr{X}$ under a determinantal point process can be calculated from (1.1). For instance, when the kernel $K$ is Hermitian and satisfies the reproducing property, i.e., it represents an orthogonal projection, then

$$\mathbb{E}\left[\sum_{x \in \mathscr{X}} f(x)\right] = \int_X f(x) K(x, x) \,\mathrm{d}\mu(x), \tag{1.2}$$

and

$$\mathrm{Var}\left(\sum_{x \in \mathscr{X}} f(x)\right) = \frac{1}{2} \int_{X^2} \left[f(x) - f(y)\right]^2 \left|K(x, y)\right|^2 \mathrm{d}\mu(x)\,\mathrm{d}\mu(y). \tag{1.3}$$

See, e.g., [6, Proposition 4.1], [7, Lemma 8.5] and [18, formulas (4) and (5)].

**1.2. The Ginibre ensemble.** Consider the Gaussian measure $\mu$ on the whole plane $\mathbb{C}$ given by

$$\mathrm{d}\mu(z) = \frac{1}{\pi} e^{-|z|^2} \mathrm{d}m(z),$$

where $\mathrm{d}m$ is the usual Lebesgue measure. We also need to consider the space $\mathcal{O}(\Lambda)$ of holomorphic functions on an open set $\Lambda \subset \mathbb{C}$.

In the finite-dimensional setting, the Ginibre ensemble was introduced by Ginibre [8] as a model based on the eigenvalues of non-Hermitian random matrices, and the infinite Ginibre ensemble is obtained as a weak limit of these finite-dimensional point processes. The infinite Ginibre ensemble can be defined as follows.

**Definition 1.1** (Ginibre ensemble)**.** The *Ginibre ensemble* $\mathcal{G}$ is the determinantal point process associated to the Fock kernel, i.e., the kernel of the orthogonal projection of $L^2(\mathbb{C},\mu)$ onto $L^2(\mathbb{C},\mu)\cap\mathcal{O}(\mathbb{C})$, with respect to the reference measure $\mu$.

Equivalently, the Ginibre ensemble is the point process $\mathcal{G}$ on $\mathbb{C}$ such that for any pairwise disjoint measurable subsets $\Delta_1,\Delta_2,\dots,\Delta_k$ of $\mathbb{C}$, we have that

$$\mathbb{E}\left[\prod_{i=1}^{k}\mathcal{G}(\Delta_i)\right]=\int_{\Delta_1\times\cdots\times\Delta_k}\det\left[K_{\mathcal{G}}(z_i,z_j)\right]_{1\le i,j\le k}\mathrm{d}m(z_1)\cdots\mathrm{d}m(z_k),$$

where $K_{\mathcal{G}}:\mathbb{C}\times\mathbb{C}\to\mathbb{C}$ is given by

$$K_{\mathcal{G}}(z,w)=\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{z^n\overline{w}^n}{n!}e^{-\frac{|z|^2}{2}}e^{-\frac{|w|^2}{2}}=\frac{1}{\pi}e^{z\overline{w}-\frac{|z|^2}{2}-\frac{|w|^2}{2}}.$$

**1.3. The hyperbolic ensembles.** For each $\alpha>0$, consider the probability measure $\mu_\alpha$ on the unit disc $\mathbb{D}$ given by

$$\mathrm{d}\mu_\alpha(z)=\frac{\alpha}{\pi}(1-|z|^2)^{\alpha-1}\mathrm{d}m(z),$$

and recall that $\mathcal{O}(\mathbb{D})$ denotes the space of holomorphic functions on $\mathbb{D}$.

**Definition 1.2** (Hyperbolic ensemble)**.** For $\alpha>0$, the *$\alpha$-hyperbolic ensemble* $\mathcal{H}_\alpha$ is the determinantal point process associated to the $\mu_\alpha$-weighted Bergman kernel, i.e., the kernel of the orthogonal projection of $L^2(\mathbb{D},\mu_\alpha)$ onto the closed subspace $L^2(\mathbb{D},\mu_\alpha)\cap\mathcal{O}(\mathbb{D})$, with respect to the reference measure $\mu_\alpha$.

Equivalently, the $\alpha$-hyperbolic ensemble is the point process $\mathcal{H}_\alpha$ on $\mathbb{D}$ such that for any pairwise disjoint measurable subsets $\Delta_1,\Delta_2,\dots,\Delta_k$ of $\mathbb{D}$,

$$\mathbb{E}\left[\prod_{i=1}^{k}\mathcal{H}_\alpha(\Delta_i)\right]=\int_{\Delta_1\times\cdots\times\Delta_k}\det\left[K_{\mathcal{H}_\alpha}(z_i,z_j)\right]_{1\le i,j\le k}\mathrm{d}m(z_1)\cdots\mathrm{d}m(z_k),$$

where $K_{\mathcal{H}_\alpha}:\mathbb{D}\times\mathbb{D}\to\mathbb{C}$ is given by

$$\begin{aligned}K_{\mathcal{H}_\alpha}(z,w)&=\frac{1}{\pi}\sum_{n=0}^{\infty}k_n^{(\alpha)}z^n\overline{w}^n(1-|z|^2)^{\frac{\alpha-1}{2}}(1-|w|^2)^{\frac{\alpha-1}{2}}\\&=\frac{\alpha}{\pi}\frac{(1-|z|^2)^{\frac{\alpha-1}{2}}(1-|w|^2)^{\frac{\alpha-1}{2}}}{(1-z\overline{w})^{\alpha+1}},\end{aligned}$$

while

$$k_n^{(\alpha)} = \frac{\alpha(\alpha+1)\cdots(\alpha+n)}{n!} = \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha)\Gamma(n+1)}.$$

For $\alpha = 1$, Peres and Virág [15] showed that the zeros of the random analytic function

$$\phi(z) = a_0 + a_1 z + a_2 z^2 + \cdots,$$

where $a_k$, $k \geq 0$, are independent and identically distributed standard complex Gaussian random variables, follow the law of the 1-hyperbolic ensemble $\mathcal{H}_1$. Krishnapur [12] extended the result of Peres and Virág to positive integer $\alpha = m$, showing that, if $G_k$, $k \geq 0$, are independent and identically distributed $m \times m$ matrices, each with independent and identically distributed standard complex Gaussian entries, then the zeros of the random analytic function

$$\Phi(z) = \det(G_0 + G_1 z + G_2 z^2 + \cdots)$$

follow the law of the $m$-hyperbolic ensemble $\mathcal{H}_m$.

**1.4. Main results.** Recall that the unit disc $\mathbb{D}$ endowed with the metric $\mathrm{d}\mu_\alpha$ is the Poincaré model for the Lobachevsky plane. Our results involve the point process formed by $\{|z| : z \in \mathcal{G}\}$ and the one formed by $\{|z|_{\mathrm{h}} : z \in \mathcal{H}_\alpha\}$, $\alpha > 0$, where

$$|z|_{\mathrm{h}} = \log \frac{1+|z|}{1-|z|}$$

is the hyperbolic distance from $z$ to the origin.

By [9, Theorem 4.7.1], which was first noticed by Kostlan [11, Lemma 1.4] in the case of a finite number of particles, the point process $\{|z| : z \in \mathcal{G}\}$ follows the law of $\{\rho_n : n \in \mathbb{N}\}$, where $(\rho_n)_{n\geq 0}$ is a family of non-negative independent random variables such that

$$\rho_n \sim \frac{2r^{2n+1}e^{-r^2}}{n!}\,\mathrm{d}r.$$

Similarly, from [9, Theorem 4.7.1], see also [12] and [15], for $\alpha > 0$, the point process $\{|z| : z \in \mathcal{H}_\alpha\}$ follows the law of $\{\rho_n^{(\alpha)} : n \in \mathbb{N}\}$, where $(\rho_n^{(\alpha)})_{n\geq 0}$ is a family of independent random variables taking values in $[0,1]$ such that

$$\rho_n^{(\alpha)} \sim 2\frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha)\Gamma(n+1)} r^{2n+1}(1-r^2)^{\alpha-1}\,\mathrm{d}r.$$

The asymptotic behavior of $\{|z| - R : z \in \mathcal{G}\}$ and of $\{|z|_{\mathrm{h}} - R : z \in \mathcal{H}_\alpha\}$, $\alpha > 0$, as $R$ goes to infinity and under different scalings is described in the theorems below. Theorem 1.3 and Theorem 1.4 deal with a convergence of the normalized process towards a Gaussian field whose covariance kernel is not the one of the white noise. Theorem 1.5 and Theorem 1.6 deal with the intermediate case of a convergence towards the white noise. Finally, the last two theorems, Theorem 1.7 and Theorem 1.8, deal with the extreme case of the convergence towards a homogeneous Poisson process.

### 1.4.1. Convergence towards a non-trivial Gaussian limit

**Theorem 1.3.** *Denote* $C_R^{(\alpha)} = \alpha e^R/8$ *for each* $\alpha > 0$*. Then for any bounded measurable and compactly supported function* $f : \mathbb{R} \to \mathbb{R}$*,*

$$\frac{1}{\sqrt{C_R^{(\alpha)}}} \sum_{z \in \mathcal{H}_\alpha} f\big(|z|_{\mathrm{h}} - R\big) - 2\sqrt{C_R^{(\alpha)}} \int_{\mathbb{R}} f(x) e^x \,\mathrm{d}x \xrightarrow[R \to +\infty]{\text{law}} \mathcal{N}\left(0, V_f^{(\alpha)}\right),$$

*where*

$$V_f^{(\alpha)} = \frac{1}{B(\alpha, \alpha+1)} \int_{\mathbb{R}^2} \big[f(x) - f(y)\big]^2 \frac{e^{(\alpha+1)(x+y)}}{(e^x + e^y)^{2\alpha+1}} \,\mathrm{d}x\,\mathrm{d}y.$$

**Theorem 1.4.** *For any bounded measurable and compactly supported function* $f : \mathbb{R} \to \mathbb{R}$*,*

$$\frac{1}{\sqrt{R}} \sum_{z \in \mathcal{G}} f\big(|z| - R\big) - 2\sqrt{R} \int_{\mathbb{R}} f(x) \,\mathrm{d}x \xrightarrow[R \to +\infty]{\text{law}} \mathcal{N}(0, V_f),$$

*where*

$$V_f = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}^2} \big[f(x) - f(y)\big]^2 e^{-(x-y)^2} \,\mathrm{d}x\,\mathrm{d}y.$$

### 1.4.2. Convergence towards white noise

**Theorem 1.5.** *Suppose that* $a_R$ *satisfies* $1 \ll a_R \ll e^R$ *as* $R \to +\infty$*. Denote* $C_R^{(\alpha)} = \alpha e^R/8$ *for each* $\alpha > 0$*. Then for any bounded measurable and compactly supported function* $f : \mathbb{R} \to \mathbb{R}$*,*

$$\sqrt{\frac{a_R}{C_R^{(\alpha)}}} \sum_{z \in \mathcal{H}_\alpha} f\big(a_R(|z|_{\mathrm{h}} - R)\big) - 2\sqrt{\frac{C_R^{(\alpha)}}{a_R}} \int_{\mathbb{R}} f(x) e^{\frac{x}{a_R}} \,\mathrm{d}x \xrightarrow[R \to +\infty]{\text{law}} \mathcal{N}\left(0, 2\int_{\mathbb{R}} f^2(x) \,\mathrm{d}x\right).$$

**Theorem 1.6.** *Suppose that* $a_R$ *satisfies* $1 \ll a_R \ll R$ *as* $R \to +\infty$*. Then for any bounded measurable and compactly supported function* $f : \mathbb{R} \to \mathbb{R}$*,*

$$\sqrt{\frac{a_R}{R}} \sum_{z \in \mathcal{G}} f\big(a_R(|z| - R)\big) - 2\sqrt{\frac{R}{a_R}} \int_{\mathbb{R}} f(x) \,\mathrm{d}x \xrightarrow[R \to +\infty]{\text{law}} \mathcal{N}\left(0, 2\int_{\mathbb{R}} f^2(x) \,\mathrm{d}x\right).$$

Theorems 1.3–1.6 will be obtained as corollaries of the central limit theorem of Soshnikov.

### 1.4.3. Convergence towards a Poisson point process

**Theorem 1.7.** *Let* $\mathcal{P}_{\alpha/4}$ *be the Poisson point process on* $\mathbb{R}$ *with constant intensity* $\alpha/4$*,* $\alpha > 0$*. Then,*

$$\big\{e^R(|z|_{\mathrm{h}} - R) : z \in \mathcal{H}_\alpha\big\} \xrightarrow[R \to +\infty]{\text{law}} \mathcal{P}_{\alpha/4}.$$

**Theorem 1.8.** *Let $\mathcal{P}_2$ be the Poisson point process on $\mathbb{R}$ with constant intensity 2. Then,*

$$\left\{R(|z|-R) : z \in \mathcal{G}\right\} \xrightarrow[R\to+\infty]{\text{law}} \mathcal{P}_2.$$

*Remark* 1.9. An analog of the theorems above is the case of an independent and identically distributed sequence $(X_i)_{i\geq 1}$ of real random variables that follows a probability distribution $\mu$. The analogue of Theorem 1.3 and Theorem 1.4 is the classical central limit theorem that tells us that, for any compactly supported function $f : \mathbb{R} \to \mathbb{R}$, we have

$$\frac{\sum_{i=1}^n f(X_i) - n\int f\,\mathrm{d}\mu}{\sqrt{n}} \xrightarrow[n\to+\infty]{\text{law}} \mathcal{N}\left(0, \frac{1}{2}\int_{\mathbb{R}^2} [f(x)-f(y)]^2\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\right).$$

If these random variables admit a density $\rho$ which is continuous at 0, we have the convergence towards a Poisson point process, the analogue of Theorem 1.7 and Theorem 1.8,

$$\{nX_i : 1 \leq i \leq n\} \xrightarrow[n\to+\infty]{\text{law}} \mathcal{P}_{\rho(0)}.$$

Notice that we can also see, at intermediate scalings, a convergence towards the white noise for the centered linear statistics.

## 2. The hyperbolic process proof of Theorem 1.3

To study the limiting behavior of the linear statistics

$$\sum_{z\in\mathcal{H}_\alpha} f\big(|z|_{\mathrm{h}} - R\big),$$

we start by understanding the asymptotics of its expected value. Later, we study its limiting variance and conclude by Soshnikov's central limit theorem [18, Theorem 1].

We prepare two lemmas that contain some bounds that we will use.

**Lemma 2.1.** *There exists two constants $C, M > 0$ such that, for every $y \geq M$ and $t \geq 0$,*

$$\left|\left(1-\frac{2}{y+1}\right)^{2\lfloor ty\rfloor+1} - e^{-4t}\right| \leq \frac{Ce^{-t/C}}{y},$$

*where $\lfloor ty\rfloor$ is the biggest integer that does not exceed $ty$.*

*Proof of Lemma 2.1.* We can use Lagrange's mean value theorem to control the difference

$$\begin{aligned}
&\left|\left(1-\frac{2}{y+1}\right)^{2\lfloor ty\rfloor+1} - e^{-4t}\right| \\
&\leq \left|\left[\left(1+\frac{1}{\frac{y+1}{2}-1}\right)^{\frac{y+1}{2}}\right]^{-\frac{2(2\lfloor ty\rfloor+1)}{y+1}} - e^{-\frac{2(2\lfloor ty\rfloor+1)}{y+1}}\right| + \left|e^{-\frac{2(2\lfloor ty\rfloor+1)}{y+1}} - e^{-4t}\right|
\end{aligned}$$

$$= \frac{2(2\lfloor ty\rfloor+1)}{y+1}\frac{1}{\xi^{\frac{2(2\lfloor ty\rfloor+1)}{y+1}+1}}\left[\left(1+\frac{1}{\frac{y+1}{2}-1}\right)^{\frac{y+1}{2}}-e\right]+e^{\eta}\left|\frac{2(2\lfloor ty\rfloor+1)}{y+1}-4t\right|,$$

where $\xi \geq e$ and $\eta \leq -A(t+1)$ for some fixed positive constant $A < 4$ and $y$ large enough. The asymptotics of each of these terms as $y$ goes to $+\infty$ gives us the result.

Now, notice that there exist two constants $C, M > 0$ such that, for every $y \geq M$ and $t \geq 0$, we have the bounds

- $\frac{2(2\lfloor ty\rfloor+1)}{y+1} \leq C(t+1)$,
- $\frac{1}{\xi^{\frac{2(2\lfloor ty\rfloor+1)}{y+1}+1}} \leq Ce^{-t/C}$,
- $\left[\left(1+\frac{1}{\frac{y+1}{2}-1}\right)^{\frac{y+1}{2}}-e\right] \leq \frac{C}{y}$,
- $e^{\eta} \leq Ce^{-t/C}$,
- $\left|\frac{2(2\lfloor ty\rfloor+1)}{y+1}-4t\right| \leq \frac{C(t+1)}{y}$.

The lemma follows by noticing that there exists $\widetilde{C} > 0$ such that, for $t \geq 0$, the inequality $(C^3+C^2)(t+1)e^{-t/C} \leq \widetilde{C}e^{-t/\widetilde{C}}$ holds. □

For the next lemma, we recall the notation

$$k_n^{(\alpha)} = \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha)\Gamma(n+1)}.$$

**Lemma 2.2.** *There exists a constant $C > 0$ such that*

$$\left|k_{\lfloor y\rfloor}^{(\alpha)} - \frac{y^{\alpha}}{\Gamma(\alpha)}\right| \leq \begin{cases} C, & y \in [0,1] \\ Cy^{\alpha-1}, & y \in [1,+\infty) \end{cases},$$

*where $\lfloor y\rfloor$ is the biggest integer that does not exceed $y$.*

*Proof of Lemma 2.2.* This is a consequence of Stirling's series that tells us that

$$k_n^{(\alpha)} = \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha)\Gamma(n+1)} = \frac{n^{\alpha}}{\Gamma(\alpha)} + O(n^{\alpha-1}) \quad \text{as } n \to \infty.$$

The piecewise inequality comes from the fact that $k_0^{(\alpha)} \neq 0$. □

With these lemmas, it is easier to understand the asymptotics of the expected value.

**2.1. Expected value calculation for Theorem 1.3.** We will show a bit more than it is needed about the expected value asymptotics. We will see that

$$\mathbb{E}\left[\sum_{z\in\mathcal{H}_{\alpha}} f\big(|z|_{\mathrm{h}}-R\big)\right] = 2C_R^{(\alpha)}\int_{\mathbb{R}} f(x)e^{x}\,\mathrm{d}x + O(1).$$

We begin by writing

$$\begin{aligned}\mathbb{E}\left[\sum_{z\in\mathcal{H}_\alpha} f\big(|z|_{\mathrm{h}}-R\big)\right] &= \sum_{n=0}^{\infty}\mathbb{E}\Big[f\big(|\rho_n^{(\alpha)}|_{\mathrm{h}}-R\big)\Big]\\ &= \sum_{n=0}^{\infty}\int_0^1 f\left(\log\frac{1+r}{1-r}-R\right)2k_n^{(\alpha)}r^{2n+1}(1-r^2)^{\alpha-1}\mathrm{d}r,\end{aligned}$$

where we recall the notation $k_n^{(\alpha)}=\frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha)\Gamma(n+1)}$. If $x=\log\frac{1+r}{1-r}-R$, we get

$$r=\frac{e^{R+x}-1}{e^{R+x}+1}\quad\text{and}\quad \mathrm{d}r=\frac{2e^{R+x}}{(e^{R+x}+1)^2}\mathrm{d}x,$$

so that

$$\begin{aligned}&\mathbb{E}\left[\sum_{z\in\mathcal{H}_\alpha} f\big(|z|_{\mathrm{h}}-R\big)\right]\\ &=4\sum_{n=0}^{\infty}\int_{-R}^{+\infty} f(x)k_n^{(\alpha)}\left(\frac{e^{R+x}-1}{e^{R+x}+1}\right)^{2n+1}\left[1-\left(\frac{e^{R+x}-1}{e^{R+x}+1}\right)^2\right]^{\alpha-1}\frac{e^{R+x}}{(e^{R+x}+1)^2}\,\mathrm{d}x\\ &=4^\alpha\int_{-R}^{+\infty}\sum_{n=0}^{\infty} f(x)k_n^{(\alpha)}\left(1-\frac{2}{e^{R+x}+1}\right)^{2n+1}\frac{e^{\alpha(R+x)}}{(e^{R+x}+1)^{2\alpha}}\,\mathrm{d}x\\ &=4^\alpha\int_{-R}^{+\infty}\sum_{n=0}^{\infty} f(x)k_n^{(\alpha)}\left(1-\frac{2}{e^{R+x}+1}\right)^{2n+1}\frac{e^{-\alpha(R+x)}}{(1+e^{-(R+x)})^{2\alpha}}\,\mathrm{d}x.\end{aligned}$$

Notice that we have interchanged the summation and integration here which is possible since $f$ is bounded and compactly supported. Therefore, by taking $n=\lfloor te^{R+x}\rfloor$, we get

$$\begin{aligned}&e^{-R}\,\mathbb{E}\left[\sum_{z\in\mathcal{H}_\alpha} f\big(|z|_{\mathrm{h}}-R\big)\right]\\ &\quad=4^\alpha\int_{-R}^{+\infty}\sum_{n=0}^{\infty} f(x)e^x k_n^{(\alpha)}\left(1-\frac{2}{e^{R+x}+1}\right)^{2n+1}\frac{e^{-\alpha(R+x)}}{\big(1+e^{-(R+x)}\big)^{2\alpha}}e^{-(R+x)}\,\mathrm{d}x\\ &\quad=4^\alpha\int_{-R}^{+\infty}\int_0^{+\infty} f(x)e^x k_{\lfloor te^{R+x}\rfloor}^{(\alpha)}\\ &\qquad\qquad\times\left(1-\frac{2}{e^{R+x}+1}\right)^{2\lfloor te^{R+x}\rfloor+1}\frac{e^{-\alpha(R+x)}}{\big(1+e^{-(R+x)}\big)^{2\alpha}}\,\mathrm{d}t\,\mathrm{d}x.\end{aligned}\tag{2.1}$$

We define the functions $\Theta_1$ and $\Theta_2$ of $(x,t,R)$ and the function $\Theta_3$ of $(x,R)$ by

$$k_{\lfloor te^{R+x}\rfloor}^{(\alpha)}=\frac{t^\alpha e^{\alpha(R+x)}}{\Gamma(\alpha)}+\Theta_1(x,t,R),\tag{2.2}$$

$$\left(1-\frac{2}{e^{R+x}+1}\right)^{2\lfloor te^{R+x}\rfloor+1}=e^{-4t}+\Theta_2(x,t,R),\tag{2.3}$$

$$\frac{1}{(1+e^{-(R+x)})^{2\alpha}} = 1 + \Theta_3(x,R). \tag{2.4}$$

By Lemma 2.2 and Lemma 2.1 (and by a standard fact for (2.4)), there exists a constant $C>0$ such that, for $R$ large enough,

$$|\Theta_1(x,t,R)| \le \begin{cases} C, & (x,t)\in \operatorname{supp} f\times[0,e^{-R}) \\ Ct^{\alpha-1}e^{(\alpha-1)R}, & (x,t)\in \operatorname{supp} f\times[e^{-R},+\infty) \end{cases},$$
$$|\Theta_2(x,t,R)| \le Ce^{-t/C}e^{-R}, \qquad (x,t)\in \operatorname{supp} f\times[0,+\infty),$$
$$|\Theta_3(x,R)| \le Ce^{-R}, \qquad x\in \operatorname{supp} f.$$

The inequality involving $\Theta_1$ deserves some explanation. Lemma 2.2 implies that for any $a,b>0$, not necessarily $a\le b$ (in fact, we will need to use it in the case $a>b$) there exists a constant $C>0$ that depends on $a$ and $b$ and such that

$$\left| k^{(\alpha)}_{\lfloor y\rfloor} - \frac{y^\alpha}{\Gamma(\alpha)}\right| \le \begin{cases} C, & y\in[0,a] \\ Cy^{\alpha-1}, & y\in[b,+\infty) \end{cases}.$$

Let $a>0$ be such that $e^x\le a$ for every $x\in\operatorname{supp} f$ and let $b>0$ be such that $b\le e^x$ for every $x\in\operatorname{supp} f$. Then, we have

$$\left| k^{(\alpha)}_{\lfloor te^{R+x}\rfloor} - \frac{(te^{R+x})^\alpha}{\Gamma(\alpha)}\right| \le \begin{cases} C, & te^{R+x}\in[0,a] \\ C(te^{R+x})^{\alpha-1}, & te^{R+x}\in[b,+\infty) \end{cases}.$$

In particular,

$$\left| k^{(\alpha)}_{\lfloor te^{R+x}\rfloor} - \frac{(te^{R+x})^\alpha}{\Gamma(\alpha)}\right| \le \begin{cases} C, & te^{R}\in[0,1] \\ C(te^{R+x})^{\alpha-1}, & te^{R}\in[1,+\infty) \end{cases}$$

whenever $x\in\operatorname{supp} f$. By bounding $(e^x)^{\alpha-1}$ by a constant for $x\in\operatorname{supp} f$ and choosing, in this way, a larger constant $C>0$, the inequality for $\Theta_1$ is obtained.

With the help of the estimates (2.2), (2.3) and (2.4), it follows from (2.1) that

$$\begin{aligned} e^{-R}\,\mathbb{E}&\left[\sum_{z\in\mathcal{H}_\alpha} f\big(|z|_{\mathrm{h}}-R\big)\right] \\ &= 4^\alpha \int_{-R}^{+\infty}\int_0^{+\infty} f(x)e^x \frac{t^\alpha e^{\alpha(R+x)}}{\Gamma(\alpha)} e^{-4t}e^{-\alpha(R+x)}\,\mathrm{d}t\,\mathrm{d}x + O\left(e^{-R}\right) \\ &= \frac{4^\alpha}{\Gamma(\alpha)}\int_0^{+\infty} t^\alpha e^{-4t}\mathrm{d}t \int_{\mathbb{R}} f(x)e^x\,\mathrm{d}x + O\left(e^{-R}\right) \\ &= \frac{\alpha}{4}\int_{\mathbb{R}} f(x)e^x\mathrm{d}x + O\left(e^{-R}\right). \end{aligned}$$

That is,

$$\mathbb{E}\Big[\sum_{z\in\mathcal{H}_\alpha} f\big(|z|_{\mathrm{h}}-R\big)\Big] = 2C_R^{(\alpha)}\int_{\mathbb{R}} f(x)e^x\mathrm{d}x + O(1).$$

**2.2. Variance calculation for Theorem 1.3.** We now turn to calculate the variance of the linear statistics $\sum_{z\in\mathcal{H}_\alpha} f\big(|z|_{\mathrm{h}}-R\big)$,

$$\mathrm{Var}\left(\sum_{z\in\mathcal{H}_\alpha} f\big(|z|_{\mathrm{h}}-R\big)\right)=\sum_{n=0}^{\infty}\mathrm{Var}\left(f\big(|\rho_n^{(\alpha)}|_{\mathrm{h}}-R\big)\right)$$
$$=\sum_{n=0}^{\infty}\mathbb{E}\Big[f^2\big(|\rho_n^{(\alpha)}|_{\mathrm{h}}-R\big)\Big]-\sum_{n=0}^{\infty}\mathbb{E}\Big[f\big(|\rho_n^{(\alpha)}|_{\mathrm{h}}-R\big)\Big]^2.$$

Due to the asymptotics in Lemma 2.1 and Lemma 2.2 we can obtain a bit more than what it is needed. For the first term $\sum_{n=0}^{\infty}\mathbb{E}\big[f^2\big(|\rho_n^{(\alpha)}|_{\mathrm{h}}-R\big)\big]$, by Subsection 2.1, we have

$$\sum_{n=0}^{\infty}\mathbb{E}\Big[f^2\big(|\rho_n^{(\alpha)}|_{\mathrm{h}}-R\big)\Big]=2C_R^{(\alpha)}\int_{\mathbb{R}}f^2(x)e^x\mathrm{d}x+O(1).$$

As for the second term $\sum_{n=0}^{\infty}\mathbb{E}\big[f\big(|\rho_n^{(\alpha)}|_{\mathrm{h}}-R\big)\big]^2$, we can study it in a similar way,

$$\sum_{n=0}^{\infty}\mathbb{E}\Big[f\big(|\rho_n^{(\alpha)}|_{\mathrm{h}}-R\big)\Big]^2$$
$$=4^{2\alpha}\sum_{n=0}^{\infty}\left(\int_{-R}^{+\infty}f(x)k_n^{(\alpha)}\left(1-\frac{2}{e^{R+x}+1}\right)^{2n+1}\frac{e^{-\alpha(R+x)}}{\big(1+e^{-(R+x)}\big)^{2\alpha}}\mathrm{d}x\right)^2.$$

Notice that the summation is outside the integration. Setting $n=\lfloor te^R\rfloor$, we have

$$e^{-R}\sum_{n=0}^{\infty}\mathbb{E}\Big[f\big(|\rho_n^{(\alpha)}|_{\mathrm{h}}-R\big)\Big]^2$$
$$=4^{2\alpha}\sum_{n=0}^{\infty}\left(\int_{-R}^{+\infty}f(x)k_n^{(\alpha)}\left(1-\frac{2}{e^{R+x}+1}\right)^{2n+1}\frac{e^{-\alpha(R+x)}}{\big(1+e^{-(R+x)}\big)^{2\alpha}}\mathrm{d}x\right)^2e^{-R}$$
$$=4^{2\alpha}\int_0^{+\infty}\left(\int_{-R}^{+\infty}f(x)k_{\lfloor te^R\rfloor}^{(\alpha)}\left(1-\frac{2}{e^{R+x}+1}\right)^{2\lfloor te^R\rfloor+1}\frac{e^{-\alpha(R+x)}}{\big(1+e^{-(R+x)}\big)^{2\alpha}}\mathrm{d}x\right)^2\mathrm{d}t.$$

We define the function $\Theta_4$ of $(t,R)$ and the function $\Theta_5$ of $(x,t,R)$ by

$$k_{\lfloor te^R\rfloor}^{(\alpha)}=\frac{t^\alpha e^{\alpha R}}{\Gamma(\alpha)}+\Theta_4(t,R),$$
$$\left(1-\frac{2}{e^{R+x}+1}\right)^{2\lfloor te^R\rfloor+1}=e^{-4te^{-x}}+\Theta_5(x,t,R),$$

so that, by Lemma 2.2 and Lemma 2.1, there exists a constant $C>0$ such that, for $R$ large enough, we have the bounds

$$|\Theta_4(t,R)|\le\begin{cases}C, & t\in[0,e^{-R})\\ Ct^{\alpha-1}e^{(\alpha-1)R}, & t\in[e^{-R},+\infty)\end{cases},$$

$$|\Theta_5(x,t,R)| \le Ce^{-t/C}e^{-R}, \quad (x,t) \in \operatorname{supp} f \times [0,+\infty).$$

With the help of these estimates and recalling the standard estimate (2.4), it follows that

$$\begin{aligned}
&e^{-R}\sum_{n=0}^{\infty}\mathbb{E}\Big[f\big(|\rho_n^{(\alpha)}|_{\mathrm{h}}-R\big)\Big]^2 \\
&= 4^{2\alpha}\int_0^{+\infty}\left(\int_{-R}^{+\infty} f(x)\frac{t^\alpha e^{\alpha R}}{\Gamma(\alpha)}e^{-4te^{-x}}e^{-\alpha(R+x)}\mathrm{d}x\right)^2\mathrm{d}t + O\left(e^{-R}\right) \\
&= \frac{4^{2\alpha}}{\Gamma^2(\alpha)}\int_{\mathbb{R}^2} f(x)f(y)e^{-\alpha(x+y)}\int_0^{+\infty} t^{2\alpha}e^{-4t(e^{-x}+e^{-y})}\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y + O\left(e^{-R}\right) \\
&= \frac{1}{4\Gamma^2(\alpha)}\int_{\mathbb{R}^2} f(x)f(y)\frac{e^{-\alpha(x+y)}}{(e^{-x}+e^{-y})^{2\alpha+1}}\int_0^{+\infty} u^{2\alpha}e^{-u}\,\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}y + O\left(e^{-R}\right) \\
&= \frac{\alpha}{4B(\alpha,\alpha+1)}\int_{\mathbb{R}^2} f(x)f(y)\frac{e^{(\alpha+1)(x+y)}}{(e^x+e^y)^{2\alpha+1}}\,\mathrm{d}x\,\mathrm{d}y + O\left(e^{-R}\right),
\end{aligned}$$

that is,

$$\sum_{n=0}^{\infty}\mathbb{E}\Big[f\big(|\rho_n^{(\alpha)}|_{\mathrm{h}}-R\big)\Big]^2 = \frac{2C_R^{(\alpha)}}{B(\alpha,\alpha+1)}\int_{\mathbb{R}^2} f(x)f(y)\frac{e^{(\alpha+1)(x+y)}}{(e^x+e^y)^{2\alpha+1}}\mathrm{d}x\mathrm{d}y + O(1).$$

Therefore, we obtain

$$\begin{aligned}
&\operatorname{Var}\left(\sum_{z\in\mathcal{H}_\alpha} f\big(|z|_{\mathrm{h}}-R\big)\right) \\
&= 2C_R^{(\alpha)}\int_{\mathbb{R}} f^2(x)e^x\,\mathrm{d}x - \frac{2C_R^{(\alpha)}}{B(\alpha,\alpha+1)}\int_{\mathbb{R}^2} f(x)f(y)\frac{e^{(\alpha+1)(x+y)}}{(e^x+e^y)^{2\alpha+1}}\,\mathrm{d}x\,\mathrm{d}y + O(1).
\end{aligned}$$

By noticing that

$$\frac{1}{B(\alpha,\alpha+1)}\int_{\mathbb{R}}\frac{e^{(\alpha+1)(x+y)}}{(e^x+e^y)^{2\alpha+1}}\,\mathrm{d}y = e^x,$$

we may conclude that

$$\begin{aligned}
&\operatorname{Var}\left(\sum_{z\in\mathcal{H}_\alpha} f\big(|z|_{\mathrm{h}}-R\big)\right) \\
&\quad = \frac{C_R^{(\alpha)}}{B(\alpha,\alpha+1)}\int_{\mathbb{R}^2}\big[f(x)-f(y)\big]^2\frac{e^{(\alpha+1)(x+y)}}{(e^x+e^y)^{2\alpha+1}}\,\mathrm{d}x\,\mathrm{d}y + O(1).
\end{aligned}$$

**2.3. Soshnikov's conditions and conclusion of the proof.** We will use the following central limit theorem, derived by Soshnikov [18, Theorem 1], to prove Theorem 1.3.

**Theorem 2.3** (The central limit theorem of Soshnikov)**.** *For $L > 0$, suppose that $\mathscr{X}_L$ is a determinantal point process on a locally compact Polish space $X_L$ with Hermitian kernel $K_L$ with respect to the reference Radon measure $\mu_L$. By slightly abusing the notation, also denote $K_L$ the associated integral operator with integral kernel $K_L$ on $L^2(X_L, \mu_L)$, that is, $K_L : L^2(X_L, \mu_L) \to L^2(X_L, \mu_L)$ is defined by*

$$K_L f(x) = \int_{X_L} K_L(x, y) f(y) d\mu_L(y), \quad f \in L^2(X_L, \mu_L),$$

*and suppose that for any pre-compact Borel set $\Delta \subset X_L$, the operator $K_L \chi_\Delta$ is trace-class, where $\chi_\Delta$ denotes the multiplication operator by the indicator of $\Delta$.*

*Let $f_L$ be a real-valued bounded measurable function on $X_L$ with compact support and consider the linear statistics $S_{f_L} = \sum_{x \in \mathscr{X}_L} f_L(x)$. If*

- $\operatorname{Var} S_{f_L} \to +\infty$ *as $L \to +\infty$,*
- $\sup_{x \in X_L} |f_L(x)| = o((\operatorname{Var} f_L)^\varepsilon)$ *as $L \to +\infty$ for any $\varepsilon > 0$ and*
- $\mathbb{E} S_{|f_L|} = O((\operatorname{Var} S_{f_L})^\delta)$ *as $L \to +\infty$ for some $\delta > 0$,*

*then the centered normalized linear statistics converges in law to the standard normal distribution, i.e.,*

$$\frac{S_{f_L} - \mathbb{E} S_{f_L}}{\sqrt{\operatorname{Var} S_{f_L}}} \xrightarrow[L \to \infty]{\text{law}} \mathcal{N}(0, 1).$$

When $R \to +\infty$, with the help of Subsection 2.1 and Subsection 2.2, we have

$$\mathbb{E}\left[\sum_{z \in \mathcal{H}_\alpha} \left|f\big(|z|_{\mathrm{h}} - R\big)\right|\right] \sim 2C_R^{(\alpha)} \int_{\mathbb{R}} |f(x)| e^x \mathrm{d}x,$$

and

$$\operatorname{Var}\left(\sum_{z \in \mathcal{H}_\alpha} f\big(|z|_{\mathrm{h}} - R\big)\right) \sim C_R^{(\alpha)} V_f^{(\alpha)}.$$

Applying Soshnikov's Theorem 2.3 to $\sum_{z \in \mathcal{H}_\alpha} f\big(|z|_{\mathrm{h}} - R\big)$, we get

$$\frac{\sum_{z \in \mathcal{H}_\alpha} f\big(|z|_{\mathrm{h}} - R\big) - \mathbb{E}\big[\sum_{z \in \mathcal{H}_\alpha} f\big(|z|_{\mathrm{h}} - R\big)\big]}{\sqrt{\operatorname{Var}\big(\sum_{z \in \mathcal{H}_\alpha} f\big(|z|_{\mathrm{h}} - R\big)\big)}} \xrightarrow[R \to +\infty]{\text{law}} \mathcal{N}(0, 1).$$

This gives that

$$\frac{1}{\sqrt{C_R^{(\alpha)}}} \sum_{z \in \mathcal{H}_\alpha} f\big(|z|_{\mathrm{h}} - R\big) - \frac{1}{\sqrt{C_R^{(\alpha)}}} \mathbb{E}\left[\sum_{z \in \mathcal{H}_\alpha} f\big(|z|_{\mathrm{h}} - R\big)\right] \xrightarrow[R \to +\infty]{\text{law}} \mathcal{N}\left(0, V_f^{(\alpha)}\right).$$

Moreover, it follows from Subsection 2.1 that

$$\frac{1}{\sqrt{C_R^{(\alpha)}}} \mathbb{E}\left[\sum_{z \in \mathcal{H}_\alpha} f\big(|z|_{\mathrm{h}} - R\big)\right] - 2\sqrt{C_R^{(\alpha)}} \int_{\mathbb{R}} f(x) e^x \, \mathrm{d}x \xrightarrow[R \to +\infty]{} 0,$$

hence

$$\frac{1}{\sqrt{C_R^{(\alpha)}}} \sum_{z\in\mathcal{H}_\alpha} f\big(|z|_{\mathrm{h}} - R\big) - 2\sqrt{C_R^{(\alpha)}} \int_{\mathbb{R}} f(x) e^x \, \mathrm{d}x \xrightarrow[R\to+\infty]{\text{law}} \mathcal{N}\left(0, V_f^{(\alpha)}\right).$$

This completes the proof of Theorem 1.3.

## 3. The Ginibre process proof of Theorem 1.4

The proof of Theorem 1.4 follows the same ideas as the proof of Theorem 1.3. Moreover, the calculation of the variance can be nicely done using the same Riemann sum's argument as before. Nevertheless, we have decided to use formulas (1.2) and (1.3) to emphasize a slightly different way.

**3.1. Expected value calculation for Theorem 1.4.** For the expectation of the random variable $\sum_{z\in\mathcal{G}} f\big(|z| - R\big)$, notice that, since $K_{\mathcal{G}}(z,z) = 1/\pi$, we have

$$\begin{aligned} \mathbb{E}\left[\sum_{z\in\mathcal{G}} f\big(|z| - R\big)\right] &= \int_{\mathbb{C}} f\big(|z| - R\big) K_{\mathcal{G}}(z,z) \, \mathrm{d}m(z) \\ &= \frac{1}{\pi} \int_{\mathbb{C}} f\big(|z| - R\big) \, \mathrm{d}m(z). \end{aligned}$$

By using polar coordinates and making a variable substitution, we get

$$\begin{aligned} \mathbb{E}\left[\sum_{z\in\mathcal{G}} f\big(|z| - R\big)\right] &= 2\int_0^{+\infty} f(r-R) r \, \mathrm{d}r \\ &= 2R \int_{-R}^{+\infty} f(x) \, \mathrm{d}x + 2\int_{-R}^{+\infty} x f(x) \, \mathrm{d}x, \end{aligned}$$

that is,

$$\mathbb{E}\left[\sum_{z\in\mathcal{G}} f\big(|z| - R\big)\right] = 2R \int_{\mathbb{R}} f(x) \mathrm{d}x + O(1).$$

**3.2. Variance calculation for Theorem 1.4.** For the variance of the random variable $\sum_{z\in\mathcal{G}} f\big(|z| - R\big)$, we use that

$$\big|K_{\mathcal{G}}(z,w)\big|^2 = \frac{1}{\pi^2} e^{-|z-w|^2},$$

to obtain

$$\begin{aligned} &\operatorname{Var}\left(\sum_{z\in\mathcal{G}} f\big(|z| - R\big)\right) \\ &\quad = \frac{1}{2} \int_{\mathbb{C}^2} \big[f\big(|z| - R\big) - f\big(|w| - R\big)\big]^2 \big|K_{\mathcal{G}}(z,w)\big|^2 \, \mathrm{d}m(z) \, \mathrm{d}m(w) \end{aligned}$$

$$= \frac{1}{2\pi^2} \int_{\mathbb{C}^2} \big[f\big(|z|-R\big) - f\big(|w|-R\big)\big]^2 e^{-|z-w|^2} \, \mathrm{d}m(z) \, \mathrm{d}m(w).$$

By using polar coordinates, we get

$$\begin{aligned}
&\mathrm{Var}\Bigg(\sum_{z\in\mathcal{G}} f\big(|z|-R\big)\Bigg) \\
&= \frac{1}{2\pi^2} \int_0^{+\infty} \int_0^{+\infty} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \big[f(r-R) - f(\rho-R)\big]^2 e^{-|re^{i\theta}-\rho e^{i\varphi}|^2} r\rho \, \mathrm{d}\theta \, \mathrm{d}\varphi \, \mathrm{d}r \, \mathrm{d}\rho \\
&= \frac{1}{\pi} \int_0^{+\infty} \int_0^{+\infty} \int_{-\pi}^{\pi} \big[f(r-R) - f(\rho-R)\big]^2 e^{-|re^{i\theta}-\rho|^2} r\rho \, \mathrm{d}\theta \, \mathrm{d}\varphi \, \mathrm{d}r \\
&= \frac{2}{\pi} \int_0^{+\infty} \int_0^{+\infty} \big[f(r-R) - f(\rho-R)\big]^2 e^{-r^2-\rho^2} r\rho \int_0^{\pi} e^{2r\rho\cos\theta} \, \mathrm{d}\theta \, \mathrm{d}r \, \mathrm{d}\rho.
\end{aligned}$$

By the change of variables $x = r - R$ and $y = \rho - R$, we have

$$\begin{aligned}
&\mathrm{Var}\Bigg(\sum_{z\in\mathcal{G}} f\big(|z|-R\big)\Bigg) \\
&= \frac{2}{\pi} \int_{-R}^{+\infty} \int_{-R}^{+\infty} \big[f(x) - f(y)\big]^2 e^{-(R+x)^2-(R+y)^2} (R+x)(R+y) \\
&\qquad \times \int_0^{\pi} e^{2(R+x)(R+y)\cos\theta} \, \mathrm{d}\theta \, \mathrm{d}x \, \mathrm{d}y \\
&= \frac{2}{\pi} \int_{-R}^{+\infty} \int_{-R}^{+\infty} \big[f(x) - f(y)\big]^2 e^{-(x-y)^2} (R+x)(R+y) \\
&\qquad \times \int_0^{\pi} e^{-4(R+x)(R+y)\sin^2\frac{\theta}{2}} \, \mathrm{d}\theta \, \mathrm{d}x \, \mathrm{d}y.
\end{aligned}$$

Setting $\theta = t/\sqrt{(R+x)(R+y)}$, we obtain

$$\begin{aligned}
\mathrm{Var}\Bigg(\sum_{z\in\mathcal{G}} f\big(|z|-R\big)\Bigg) &= \frac{2}{\pi} \int_{-R}^{+\infty} \int_{-R}^{+\infty} \big[f(x) - f(y)\big]^2 e^{-(x-y)^2} \sqrt{(R+x)(R+y)} \\
&\qquad \times \int_0^{\pi\sqrt{(R+x)(R+y)}} e^{-4(R+x)(R+y)\sin^2\frac{t}{2\sqrt{(R+x)(R+y)}}} \, \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}y.
\end{aligned}$$

Consider the integral

$$\begin{aligned}
&\int_{\mathbb{R}^3} \big[f(x) - f(y)\big]^2 e^{-(x-y)^2} \sqrt{(1+x/R)(1+y/R)} \, e^{-4(R+x)(R+y)\sin^2\frac{t}{2\sqrt{(R+x)(R+y)}}} \\
&\qquad \times \chi_{\left[0,\pi\sqrt{(R+x)(R+y)}\right]}(t) \chi_{[-R,+\infty)}(x) \chi_{[-R,+\infty)}(y) \, \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}y.
\end{aligned}$$

When $R$ is sufficiently large, the integrand is dominated by

$$\big[f(x) - f(y)\big]^2 e^{-(x-y)^2} \sqrt{(1+|x|)(1+|y|)} \, e^{-\frac{4}{\pi^2}t^2} \chi_{[0,+\infty]}(t),$$

hence by the dominated convergence theorem, as $R \to +\infty$, the integral converges to

$$\frac{\sqrt{\pi}}{2} \int_{\mathbb{R}^2} \left[f(x) - f(y)\right]^2 e^{-(x-y)^2} \, \mathrm{d}x \, \mathrm{d}y.$$

It follows that, when $R \to +\infty$,

$$\operatorname{Var}\left(\sum_{z \in \mathcal{G}} f\big(|z| - R\big)\right) \sim \frac{R}{\sqrt{\pi}} \int_{\mathbb{R}^2} \left[f(x) - f(y)\right]^2 e^{-(x-y)^2} \, \mathrm{d}x \, \mathrm{d}y.$$

**3.3. Soshnikov's conditions and conclusion of the proof.** When $R \to +\infty$, with the help of Subsection 3.1 and Subsection 3.2, we have

$$\mathbb{E}\left[\sum_{z \in \mathcal{G}} \left|f\big(|z| - R\big)\right|\right] \sim 2R \int_{\mathbb{R}} |f(x)| \, \mathrm{d}x,$$

and

$$\operatorname{Var}\left(\sum_{z \in \mathcal{G}} f\big(|z| - R\big)\right) \sim R V_f.$$

Applying Soshnikov's Theorem 2.3 to $\sum_{z \in \mathcal{G}} f\big(|z| - R\big)$, we get

$$\frac{\sum_{z \in \mathcal{G}} f\big(|z| - R\big) - \mathbb{E}\left[\sum_{z \in \mathcal{G}} f\big(|z| - R\big)\right]}{\sqrt{\operatorname{Var}\left(\sum_{z \in \mathcal{G}} f\big(|z| - R\big)\right)}} \xrightarrow[R \to +\infty]{\text{law}} \mathcal{N}(0, 1).$$

This gives that

$$\frac{1}{\sqrt{R}} \sum_{z \in \mathcal{G}} f\big(|z| - R\big) - \frac{1}{\sqrt{R}} \mathbb{E}\left[\sum_{z \in \mathcal{G}} f\big(|z| - R\big)\right] \xrightarrow[R \to +\infty]{\text{law}} \mathcal{N}(0, V_f).$$

Moreover, it follows from Subsection 3.1 that

$$\frac{1}{\sqrt{R}} \mathbb{E}\left[\sum_{z \in \mathcal{G}} f\big(|z| - R\big)\right] - 2\sqrt{R} \int_{\mathbb{R}} f(x) \, \mathrm{d}x \xrightarrow[R \to +\infty]{} 0,$$

hence

$$\frac{1}{\sqrt{R}} \sum_{z \in \mathcal{G}} f\big(|z| - R\big) - 2\sqrt{R} \int_{\mathbb{R}} f(x) \, \mathrm{d}x \xrightarrow[R \to +\infty]{\text{law}} \mathcal{N}(0, V_f).$$

This completes the proof of Theorem 1.4.

## 4. Proof of Theorem 1.5 (hyperbolic case)

In the case $1 \ll a_R \ll e^R$ as $R \to +\infty$, the proof of Theorem 1.5 follows the same steps as the proof of Theorem 1.3: first we need to understand the expected value, then to calculate the limiting variance, and finally to use Soshnikov's Theorem 2.3.

Let us begin by writing

$$\begin{aligned}
&\mathbb{E}\left[\sum_{z\in\mathcal{H}_\alpha} f\big(a_R(|z|_{\mathrm{h}}-R)\big)\right]\\
&= 4^\alpha \int_{-R}^{+\infty} \sum_{n=0}^{\infty} f(a_R y) k_n^{(\alpha)} \left(1-\frac{2}{e^{R+y}+1}\right)^{2n+1} \frac{e^{-\alpha(R+y)}}{(1+e^{-(R+y)})^{2\alpha}}\,\mathrm{d}y\\
&= 4^\alpha \int_{-Ra_R}^{+\infty} \sum_{n=0}^{\infty} f(x) k_n^{(\alpha)} \left(1-\frac{2}{e^{R+\frac{x}{a_R}}+1}\right)^{2n+1} \frac{e^{-\alpha(R+\frac{x}{a_R})}}{\big(1+e^{-(R+\frac{x}{a_R})}\big)^{2\alpha}}\frac{\mathrm{d}x}{a_R}.
\end{aligned}$$

By taking $n = \lfloor t e^{R+x/a_R}\rfloor$, we get

$$\begin{aligned}
&\frac{a_R}{e^R}\,\mathbb{E}\left[\sum_{z\in\mathcal{H}_\alpha} f\big(a_R(|z|_{\mathrm{h}}-R)\big)\right]\\
&\quad= 4^\alpha \int_{-Ra_R}^{+\infty} \sum_{n=0}^{\infty} f(x) e^{\frac{x}{a_R}} k_n^{(\alpha)} \left(1-\frac{2}{e^{R+\frac{x}{a_R}}+1}\right)^{2n+1}\\
&\qquad\qquad\times \frac{e^{-\alpha\left(R+\frac{x}{a_R}\right)}}{\left(1+e^{-\left(R+\frac{x}{a_R}\right)}\right)^{2\alpha}} e^{-\left(R+\frac{x}{a_R}\right)}\,\mathrm{d}x\\
&\quad= 4^\alpha \int_{-Ra_R}^{+\infty}\int_0^{+\infty} f(x) e^{\frac{x}{a_R}} k^{(\alpha)}_{\left\lfloor t e^{R+\frac{x}{a_R}}\right\rfloor} \left(1-\frac{2}{e^{R+\frac{x}{a_R}}+1}\right)^{2\left\lfloor t e^{R+\frac{x}{a_R}}\right\rfloor+1}\\
&\qquad\qquad\times \frac{e^{-\alpha\left(R+\frac{x}{a_R}\right)}}{\big(1+e^{-\left(R+\frac{x}{a_R}\right)}\big)^{2\alpha}}\,\mathrm{d}t\,\mathrm{d}x.
\end{aligned}$$

This can be done in the same way as in Subsection 2.1. Notice that $x \in \operatorname{supp} f$ and $1 \ll a_R \ll e^R$ as $R \to +\infty$. We can show that for sufficiently large $R$,

$$\begin{aligned}
&\frac{a_R}{e^R}\mathbb{E}\left[\sum_{z\in\mathcal{H}_\alpha} f\big(a_R(|z|_{\mathrm{h}}-R)\big)\right]\\
&\quad= 4^\alpha \int_{-Ra_R}^{+\infty}\int_0^{+\infty} f(x) e^{\frac{x}{a_R}} \frac{t^\alpha e^{\alpha(R+\frac{x}{a_R})}}{\Gamma(\alpha)} e^{-4t} e^{-\alpha(R+\frac{x}{a_R})}\,\mathrm{d}t\,\mathrm{d}x + O\left(e^{-R}\right)\\
&\quad= \frac{4^\alpha}{\Gamma(\alpha)}\int_0^{+\infty} t^\alpha e^{-4t}\,\mathrm{d}t \int_{\mathbb{R}} f(x) e^{\frac{x}{a_R}}\,\mathrm{d}x + O\left(e^{-R}\right)
\end{aligned}$$

$$= \frac{\alpha}{4}\int_{\mathbb{R}} f(x)e^{\frac{x}{a_R}}\,\mathrm{d}x + O\left(e^{-R}\right),$$

that is,

$$\mathbb{E}\left[\sum_{z\in\mathcal{H}_\alpha} f\big(a_R(|z|_{\mathrm{h}}-R)\big)\right] = \frac{2C_R^{(\alpha)}}{a_R}\int_{\mathbb{R}} f(x)e^{\frac{x}{a_R}}\,\mathrm{d}x + O\left(\frac{1}{a_R}\right).$$

As for the variance

$$\operatorname{Var}\left(\sum_{z\in\mathcal{H}_\alpha} f\big(a_R(|z|_{\mathrm{h}}-R)\big)\right) = \sum_{n=0}^{\infty}\mathbb{E}\Big[f^2\big(a_R(|\rho_n^{(\alpha)}|_{\mathrm{h}}-R)\big)\Big] - \sum_{n=0}^{\infty}\mathbb{E}\Big[f\big(a_R(|\rho_n^{(\alpha)}|_{\mathrm{h}}-R)\big)\Big]^2,$$

the first term satisfies

$$\sum_{n=0}^{\infty}\mathbb{E}\Big[f^2\big(a_R(|\rho_n^{(\alpha)}|_{\mathrm{h}}-R)\big)\Big] = \frac{2C_R^{(\alpha)}}{a_R}\int_{\mathbb{R}} f^2(x)e^{\frac{x}{a_R}}\,\mathrm{d}x + O\left(\frac{1}{a_R}\right) = \frac{2C_R^{(\alpha)}}{a_R}\int_{\mathbb{R}} f^2(x)\,\mathrm{d}x + O\left(\frac{C_R^{(\alpha)}}{a_R^2}\right).$$

The second term can be dealt with in the same way as in Subsection 2.2,

$$\sum_{n=0}^{\infty}\mathbb{E}\Big[f\big(a_R(|\rho_n^{(\alpha)}|_{\mathrm{h}}-R)\big)\Big]^2 = 4^{2\alpha}\sum_{n=0}^{\infty}\left(\int_{-R}^{+\infty} f(a_R x)k_n^{(\alpha)}\left(1-\frac{2}{e^{R+x}+1}\right)^{2n+1}\frac{e^{\alpha(R+x)}}{(e^{R+x}+1)^{2\alpha}}\mathrm{d}x\right)^2 = \frac{4^{2\alpha}}{a_R^2}\sum_{n=0}^{\infty}\left(\int_{-Ra_R}^{+\infty} f(x)k_n^{(\alpha)}\left(1-\frac{2}{e^{R+\frac{x}{a_R}}+1}\right)^{2n+1}\frac{e^{\alpha\left(R+\frac{x}{a_R}\right)}}{\left(e^{R+\frac{x}{a_R}}+1\right)^{2\alpha}}\mathrm{d}x\right)^2,$$

set $n=\lfloor te^R\rfloor$, then

$$\sum_{n=0}^{\infty}\mathbb{E}\Big[f\big(a_R(|\rho_n^{(\alpha)}|_{\mathrm{h}}-R)\big)\Big]^2 = \frac{4^{2\alpha}e^R}{a_R^2}\int_0^{+\infty}\left(\int_{-Ra_R}^{+\infty} f(x)k_{\lfloor te^R\rfloor}^{(\alpha)}\left(1-\frac{2}{e^{R+\frac{x}{a_R}}+1}\right)^{2\lfloor te^R\rfloor+1} \times\frac{e^{\alpha\left(R+\frac{x}{a_R}\right)}}{\left(e^{R+\frac{x}{a_R}}+1\right)^{2\alpha}}\,\mathrm{d}x\right)^2\mathrm{d}t.$$

Hence when $R \to +\infty$,

$$\begin{aligned}
&\sum_{n=0}^{\infty} \mathbb{E}\Big[f\big(a_R(|\rho_n^{(\alpha)}|_{\mathrm{h}} - R)\big)\Big]^2 \\
&\quad \sim \frac{4^{2\alpha}\, e^R}{a_R^2} \int_0^{+\infty} \left( \int_{-Ra_R}^{+\infty} f(x) \frac{t^\alpha e^{\alpha R}}{\Gamma(\alpha)} e^{-4t} e^{-2\alpha R} e^{\alpha R} \,\mathrm{d}x \right)^2 \mathrm{d}t \\
&\quad = \frac{4^{2\alpha}\, e^R}{\Gamma^2(\alpha)\, a_R^2} \int_0^{+\infty} t^{2\alpha} e^{-8t} \mathrm{d}t \left( \int_{\mathbb{R}} f(x)\,\mathrm{d}x \right)^2 \\
&\quad = \frac{e^R \alpha}{a_R^2 2^{2\alpha+3} B(\alpha, \alpha+1)} \left( \int_{\mathbb{R}} f(x)\,\mathrm{d}x \right)^2 = O\left( \frac{C_R^{(\alpha)}}{a_R^2} \right).
\end{aligned}$$

Therefore, we obtain

$$\mathrm{Var}\left( \sum_{z \in \mathcal{H}_\alpha} f\big(a_R(|z|_{\mathrm{h}} - R)\big) \right) = \frac{2C_R^{(\alpha)}}{a_R} \int_{\mathbb{R}} f^2(x)\,\mathrm{d}x + O\left( \frac{C_R^{(\alpha)}}{a_R^2} \right).$$

In the case $1 \ll a_R \ll e^R$ as $R \to +\infty$, applying Soshnikov's Theorem 2.3, the above calculations yield that

$$\frac{\sum_{z \in \mathcal{H}_\alpha} f\big(a_R(|z|_{\mathrm{h}} - R)\big) - \mathbb{E}\big[\sum_{z \in \mathcal{H}_\alpha} f\big(a_R(|z|_{\mathrm{h}} - R)\big)\big]}{\sqrt{\mathrm{Var}\left(\sum_{z \in \mathcal{H}_\alpha} f\big(a_R(|z|_{\mathrm{h}} - R)\big)\right)}} \xrightarrow[R \to +\infty]{\text{law}} \mathcal{N}(0,1).$$

This gives that

$$\begin{aligned}
&\sqrt{\frac{a_R}{C_R^{(\alpha)}}} \sum_{z \in \mathcal{H}_\alpha} f\big(a_R(|z|_{\mathrm{h}} - R)\big) - \sqrt{\frac{a_R}{C_R^{(\alpha)}}} \mathbb{E}\left[ \sum_{z \in \mathcal{H}_\alpha} f\big(a_R(|z|_{\mathrm{h}} - R)\big) \right] \\
&\qquad \xrightarrow[R \to +\infty]{\text{law}} \mathcal{N}\left(0, 2\int_{\mathbb{R}} f^2(x)\,\mathrm{d}x\right).
\end{aligned}$$

Moreover,

$$\sqrt{\frac{a_R}{C_R^{(\alpha)}}} \mathbb{E}\left[ \sum_{z \in \mathcal{H}_\alpha} f\big(a_R(|z|_{\mathrm{h}} - R)\big) \right] - 2\sqrt{\frac{C_R^{(\alpha)}}{a_R}} \int_{\mathbb{R}} f(x) e^{\frac{x}{a_R}}\,\mathrm{d}x \xrightarrow[R \to +\infty]{} 0,$$

hence

$$\begin{aligned}
&\sqrt{\frac{a_R}{C_R^{(\alpha)}}} \sum_{z \in \mathcal{H}_\alpha} f\big(a_R(|z|_{\mathrm{h}} - R)\big) - 2\sqrt{\frac{C_R^{(\alpha)}}{a_R}} \int_{\mathbb{R}} f(x) e^{\frac{x}{a_R}}\,\mathrm{d}x \\
&\qquad \xrightarrow[R \to +\infty]{\text{law}} \mathcal{N}\left(0, 2\int_{\mathbb{R}} f^2(x)\mathrm{d}x\right),
\end{aligned}$$

This completes the proof of Theorem 1.5.

## 5. Proof of Theorem 1.6 (Ginibre case)

In the case $1 \ll a_R \ll R$ as $R \to +\infty$, the proof of Theorem 1.6 follows the same steps as the proof of Theorem 1.4.

For the expectation of the random variable $\sum_{z\in\mathcal{G}} f\big(a_R(|z|-R)\big)$, we have

$$\begin{aligned}\mathbb{E}\left[\sum_{z\in\mathcal{G}} f\big(a_R(|z|-R)\big)\right] &= 2R\int_{-R}^{+\infty} f(a_R x)\,\mathrm{d}x + 2\int_{-R}^{+\infty} x f(a_R x)\,\mathrm{d}x \\ &= \frac{2R}{a_R}\int_{-Ra_R}^{+\infty} f(x)\,\mathrm{d}x + \frac{2}{a_R^2}\int_{-Ra_R}^{+\infty} x f(x)\,\mathrm{d}x,\end{aligned}$$

that is,

$$\mathbb{E}\left[\sum_{z\in\mathcal{G}} f\big(a_R(|z|-R)\big)\right] = \frac{2R}{a_R}\int_{\mathbb{R}} f(x)\,\mathrm{d}x + O\left(\frac{1}{a_R^2}\right).$$

As for the variance of the random variable $\sum_{z\in\mathcal{G}} f\big(a_R(|z|-R)\big)$, we have

$$\begin{aligned}&\operatorname{Var}\left(\sum_{z\in\mathcal{G}} f\big(a_R(|z|-R)\big)\right) \\ &= \frac{2}{\pi}\int_{-R}^{+\infty}\int_{-R}^{+\infty}\big[f(a_R x)-f(a_R y)\big]^2 e^{-(x-y)^2}\sqrt{(R+x)(R+y)} \\ &\quad\times\int_0^{\pi\sqrt{(R+x)(R+y)}} e^{-4(R+x)(R+y)\sin^2\frac{t}{2\sqrt{(R+x)(R+y)}}}\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{2}{\pi a_R^2}\int_{-Ra_R}^{+\infty}\int_{-Ra_R}^{+\infty}\big[f(x)-f(y)\big]^2 e^{-\frac{(x-y)^2}{a_R^2}}\sqrt{\left(R+\frac{x}{a_R}\right)\left(R+\frac{y}{a_R}\right)} \\ &\quad\times\int_0^{\pi\sqrt{\left(R+\frac{x}{a_R}\right)\left(R+\frac{y}{a_R}\right)}} e^{-4\left(R+\frac{x}{a_R}\right)\left(R+\frac{y}{a_R}\right)\sin^2\frac{t}{2\sqrt{\left(R+\frac{x}{a_R}\right)\left(R+\frac{y}{a_R}\right)}}}\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y \\ &= I_1(R) - I_2(R),\end{aligned}$$

where

$$\begin{aligned}I_1(R) &= \frac{4R}{\pi a_R^2}\int_{-Ra_R}^{+\infty}\int_{-Ra_R}^{+\infty}\int_0^{\pi\sqrt{\left(R+\frac{x}{a_R}\right)\left(R+\frac{y}{a_R}\right)}} f^2(x)e^{-\frac{(x-y)^2}{a_R^2}} \\ &\quad\times\sqrt{\left(1+\frac{x}{Ra_R}\right)\left(1+\frac{y}{Ra_R}\right)} \\ &\quad\times e^{-4\left(R+\frac{x}{a_R}\right)\left(R+\frac{y}{a_R}\right)\sin^2\frac{t}{2\sqrt{\left(R+\frac{x}{a_R}\right)\left(R+\frac{y}{a_R}\right)}}}\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{4R}{\pi a_R}\int_{-R}^{+\infty}\int_{-Ra_R}^{+\infty}\int_0^{\pi\sqrt{\left(R+\frac{x}{a_R}\right)(R+u)}} f^2(x)e^{-\left(\frac{x}{a_R}-u\right)^2}\end{aligned}$$

$$\times \sqrt{\left(1+\frac{x}{Ra_R}\right)\left(1+\frac{u}{R}\right)}$$
$$\times e^{-4(R+\frac{x}{a_R})(R+u)\sin^2\frac{t}{2\sqrt{(R+\frac{x}{a_R})(R+u)}}}\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}u,$$

and

$$I_2(R)=\frac{4}{\pi}\frac{4R}{\pi a_R^2}\int_{-Ra_R}^{+\infty}\int_{-Ra_R}^{+\infty}\int_0^{\pi\sqrt{\left(R+\frac{x}{a_R}\right)\left(R+\frac{y}{a_R}\right)}} f(x)f(y)e^{-\frac{(x-y)^2}{a_R^2}}$$
$$\times \sqrt{\left(1+\frac{x}{Ra_R}\right)\left(1+\frac{y}{Ra_R}\right)}$$
$$\times e^{-4\left(R+\frac{x}{a_R}\right)\left(R+\frac{y}{a_R}\right)\sin^2\frac{t}{2\sqrt{\left(R+\frac{x}{a_R}\right)\left(R+\frac{y}{a_R}\right)}}}\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y.$$

Consider the integral

$$\int_{\mathbb{R}^3} f^2(x)e^{-\left(\frac{x}{a_R}-u\right)^2}\sqrt{\left(1+\frac{x}{Ra_R}\right)\left(1+\frac{u}{R}\right)}$$
$$\times e^{-4\left(R+\frac{x}{a_R}\right)(R+u)\sin^2\frac{t}{2\sqrt{\left(R+\frac{x}{a_R}\right)(R+u)}}}$$
$$\times \chi_{\left[0,\pi\sqrt{\left(R+\frac{x}{a_R}\right)(R+u)}\right]}(t)\chi_{[-Ra_R,+\infty)}(x)\chi_{[-R,+\infty)}(u)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}u.$$

When $R$ is sufficiently large, the integrand is dominated by

$$e^{2l^2}f^2(x)e^{-(|u|-l)^2}\sqrt{(1+|x|)(1+|u|)}\,e^{-\frac{4}{\pi^2}t^2}\chi_{[0,+\infty)}(t),$$

where $l=\max\limits_{x\in\operatorname{supp} f}|x|$. Here we used the fact that when $x\in\operatorname{supp} f$,

$$e^{-\left(\frac{x}{a_R}-u\right)^2}=e^{-u^2+2u\frac{x}{a_R}-\frac{x^2}{a_R^2}}\le e^{-u^2+2|u||x|+|x|^2}\le e^{-u^2+2|u|l+l^2}=e^{2l^2}e^{-(|u|-l)^2}.$$

Hence by the dominated convergence theorem, as $R\to+\infty$, the integral converges to

$$\int_{\mathbb{R}^3} f^2(x)e^{-u^2}e^{-t^2}\chi_{[0,+\infty)}(t)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}u=\frac{\pi}{2}\int_{\mathbb{R}} f^2(x)\,\mathrm{d}x.$$

It follows that when $R\to+\infty$,

$$I_1(R)\sim\frac{2R}{a_R}\int_{\mathbb{R}} f^2(x)\,\mathrm{d}x.$$

Consider the integral

$$\int_{\mathbb{R}^3} f(x)f(y)e^{-\frac{(x-y)^2}{a_R^2}}\sqrt{\left(1+\frac{x}{Ra_R}\right)\left(1+\frac{y}{Ra_R}\right)}$$

$$\times e^{-4\left(R+\frac{x}{a_R}\right)\left(R+\frac{y}{a_R}\right)\sin^2\frac{t}{2\sqrt{\left(R+\frac{x}{a_R}\right)\left(R+\frac{y}{a_R}\right)}}}$$
$$\times \chi_{\left[0,\pi\sqrt{\left(R+\frac{x}{a_R}\right)\left(R+\frac{y}{a_R}\right)}\right]}(t)\chi_{[-Ra_R,+\infty)}(x)\chi_{[-Ra_R,+\infty)}(u)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y.$$

When $R$ is sufficiently large, the integrand is dominated by

$$|f(x)f(y)|\sqrt{(1+|x|)(1+|y|)}\,e^{-\frac{4}{\pi^2}t^2}\chi_{[0,+\infty)}(t).$$

Hence by the dominated convergence theorem, as $R\to+\infty$, the integral converges to

$$\int_{\mathbb{R}^3} f(x)f(y)e^{-t^2}\chi_{[0,+\infty)}(t)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y=\frac{\sqrt{\pi}}{2}\left(\int_{\mathbb{R}} f(x)\,\mathrm{d}x\right)^2.$$

It follows that when $R\to+\infty$,

$$I_2(R)\sim\frac{2R}{\sqrt{\pi}a_R^2}\left(\int_{\mathbb{R}} f(x)\,\mathrm{d}x\right)^2=o\left(\frac{R}{a_R}\right).$$

Therefore, when $R\to+\infty$,

$$\operatorname{Var}\left(\sum_{z\in\mathcal{G}} f\big(a_R(|z|-R)\big)\right)\sim\frac{2R}{a_R}\int_{\mathbb{R}} f^2(x)\,\mathrm{d}x.$$

In the case $1\ll a_R\ll R$ as $R\to+\infty$, applying Soshnikov's Theorem 2.3, the above calculations yield that

$$\frac{\sum_{z\in\mathcal{G}} f\big(a_R(|z|-R)\big)-\mathbb{E}\big[\sum_{z\in\mathcal{G}} f\big(a_R(|z|-R)\big)\big]}{\sqrt{\operatorname{Var}\big(\sum_{z\in\mathcal{G}} f\big(a_R(|z|-R)\big)\big)}}\xrightarrow[R\to+\infty]{\text{law}}\mathcal{N}(0,1).$$

This gives that

$$\sqrt{\frac{a_R}{R}}\sum_{z\in\mathcal{G}} f\big(a_R(|z|-R)\big)-\sqrt{\frac{a_R}{R}}\mathbb{E}\left[\sum_{z\in\mathcal{G}} f\big(a_R(|z|-R)\big)\right]$$
$$\xrightarrow[R\to+\infty]{\text{law}}\mathcal{N}\left(0,2\int_{\mathbb{R}} f^2(x)\,\mathrm{d}x\right).$$

Moreover,

$$\sqrt{\frac{a_R}{R}}\mathbb{E}\left[\sum_{z\in\mathcal{G}} f\big(a_R(|z|-R)\big)\right]-2\sqrt{\frac{R}{a_R}}\int_{\mathbb{R}} f(x)\,\mathrm{d}x\xrightarrow[R\to+\infty]{}0,$$

hence

$$\sqrt{\frac{a_R}{R}}\sum_{z\in\mathcal{G}} f\big(a_R(|z|-R)\big)-2\sqrt{\frac{R}{a_R}}\int_{\mathbb{R}} f(x)\,\mathrm{d}x$$
$$\xrightarrow[R\to+\infty]{\text{law}}\mathcal{N}\left(0,2\int_{\mathbb{R}} f^2(x)\,\mathrm{d}x\right).$$

This completes the proof of Theorem 1.6.

## 6. Proof of Theorem 1.7 and Theorem 1.8

**6.1. A Lemma of convergence towards the Poisson point process.** The main reason why a Poisson point process appears is because we are dealing with independent particles. Nevertheless, this is not enough. We need that each particle escapes every bounded set and we need this to be done in a uniform way. More precisely, we use the following lemma.

**Lemma 6.1.** *For each $R > 0$, let $\{X_n^{(R)} : n \in \mathbb{N}\}$ be a sequence of independent real-valued random variables. Suppose that for every compact $K \subset \mathbb{R}$,*

$$\sup_{n\geq 0} \mathbb{P}\left(X_n^{(R)} \in K\right) \xrightarrow[R\to+\infty]{} 0,$$

*and suppose that there is a positive Radon measure $\nu$ on $\mathbb{R}$ such that*

$$\sum_{n=0}^{\infty} \mathbb{E}\left[f\left(X_n^{(R)}\right)\right] \xrightarrow[R\to+\infty]{} \int_{\mathbb{R}} f \,\mathrm{d}\nu$$

*for every measurable compactly supported function $f : \mathbb{R} \to [0, 1)$. Then for every measurable compactly supported function $f : \mathbb{R} \to [0, 1)$,*

$$\mathbb{E}\left[\prod_{n=0}^{\infty}\left(1 - f\left(X_n^{(R)}\right)\right)\right] \xrightarrow[R\to+\infty]{} \exp\left(-\int_{\mathbb{R}} f \,\mathrm{d}\nu\right).$$

*Proof.* For every measurable compactly supported function $f : \mathbb{R} \to [0, 1)$, by independence, we have

$$\mathbb{E}\left[\prod_{n=0}^{\infty}\left(1 - f\left(X_n^{(R)}\right)\right)\right] = \prod_{n=0}^{\infty}\mathbb{E}\left[1 - f\left(X_n^{(R)}\right)\right] = \prod_{n=0}^{\infty}\left(1 - \mathbb{E}\left[f\left(X_n^{(R)}\right)\right]\right),$$

and then

$$\log\mathbb{E}\left[\prod_{n=0}^{\infty}\left(1 - f\left(X_n^{(R)}\right)\right)\right] = \sum_{n=0}^{\infty}\log\left(1 - \mathbb{E}\left[f\left(X_n^{(R)}\right)\right]\right).$$

We can use the fact that $\log(1 - x) = -x + \Theta(x)$, where $\Theta(x) = O(x^2)$ as $x \to 0$, to obtain that

$$\sum_{n=0}^{\infty}\log\left(1 - \mathbb{E}\left[f\left(X_n^{(R)}\right)\right]\right) = -\sum_{n=0}^{\infty}\mathbb{E}\left[f\left(X_n^{(R)}\right)\right] + \sum_{n=0}^{\infty}\Theta\left(\mathbb{E}\left[f\left(X_n^{(R)}\right)\right]\right).$$

Note that

$$\sup_{n\geq 0}\mathbb{E}\left[f\left(X_n^{(R)}\right)\right] \leq \sup_{n\geq 0}\mathbb{P}\left(X_n^{(R)} \in \operatorname{supp} f\right) \xrightarrow[R\to+\infty]{} 0,$$

and the convergence of $\sum_{n=0}^{\infty}\mathbb{E}\left[f\left(X_n^{(R)}\right)\right]$, we get

$$\sum_{n=0}^{\infty}\Theta\left(\mathbb{E}\left[f\left(X_n^{(R)}\right)\right]\right) = O\left(\sum_{n=0}^{\infty}\mathbb{E}\left[f\left(X_n^{(R)}\right)\right]^2\right)$$

$$= O\left(\sup_{n\geq 0}\mathbb{E}\left[f\left(X_n^{(R)}\right)\right]\right) \xrightarrow[R\to+\infty]{} 0.$$

This implies that

$$\sum_{n=0}^{\infty}\log\left(1-\mathbb{E}\left[f\left(X_n^{(R)}\right)\right]\right) \xrightarrow[R\to+\infty]{} -\int_{\mathbb{R}} f\,\mathrm{d}\nu.$$

Therefore,

$$\mathbb{E}\left[\prod_{n=0}^{\infty}\left(1-f(X_n^{(R)})\right)\right] \xrightarrow[R\to+\infty]{} \exp\left(-\int_{\mathbb{R}} f\,\mathrm{d}\nu\right).$$

This completes the proof of Lemma 6.1. □

**Corollary 6.2** (Convergence towards a PPP)**.** *Under the conditions of Lemma 6.1. If, moreover, $\mathcal{X}^{(R)} = \{X_n^{(R)} : n \in \mathbb{N}\}$ is a point process on $\mathbb{R}$ for each $R > 0$. Then*

$$\mathcal{X}^{(R)} \xrightarrow[R\to+\infty]{\text{law}} \mathcal{P}_\nu,$$

*where $\mathcal{P}_\nu$ is the Poisson point process on $\mathbb{R}$ with mean measure or intensity $\nu$.*

**6.2. Proof of Theorem 1.7 (hyperbolic case).** Recall that $(\rho_n^{(\alpha)})_{n\geq 0}$, $\alpha > 0$, is a family of non-negative independent random variables such that

$$\rho_n^{(\alpha)} \sim 2\frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha)\Gamma(n+1)} r^{2n+1}(1-r^2)^{\alpha-1}\,\mathrm{d}r.$$

Fix $\alpha > 0$, if we define $X_n^{(R)} = e^R(|\rho_n^{(\alpha)}|_{\mathrm{h}} - R)$ for each $R > 0$ and $n \in \mathbb{N}$, we can obtain that

$$\left\{e^R(|z|_{\mathrm{h}} - R) : z \in \mathcal{H}_\alpha\right\} \sim \{X_n^{(R)} : n \in \mathbb{N}\}.$$

We are going to show that

$$\left\{X_n^{(R)} : n \in \mathbb{N}\right\} \xrightarrow[R\to+\infty]{\text{law}} \mathcal{P}_{\alpha/4}.$$

For every measurable compactly supported function $f : \mathbb{R} \to [0, 1)$, a similar argument as Section 4 gives that

$$\begin{aligned}\sum_{n=0}^{\infty}\mathbb{E}\left[f\left(X_n^{(R)}\right)\right] = 4^\alpha \int_{-Re^R}^{+\infty}\int_0^{+\infty} & f(x)e^{x/e^R}k^{(\alpha)}_{\lfloor te^{R+x/e^R}\rfloor} \\ \times \left(1-\frac{2}{e^{R+x/e^R}+1}\right)^{2\lfloor te^{R+x/e^R}\rfloor+1} & \frac{e^{\alpha(R+x/e^R)}}{(e^{R+x/e^R}+1)^{2\alpha}}\,\mathrm{d}t\,\mathrm{d}x \\ & \xrightarrow[R\to+\infty]{} \frac{\alpha}{4}\int_{\mathbb{R}} f(x)\,\mathrm{d}x.\end{aligned}$$

Hence by Corollary 6.2, we shall prove that for every $T > 0$,

$$\sup_{n\geq 0} \mathbb{P}\left(X_n^{(R)} \in [-T, T]\right) \xrightarrow[R\to+\infty]{} 0.$$

When $R$ is large enough, we have

$$\begin{aligned}
&\mathbb{P}(X_n^{(R)} \in [-T, T]) \\
&= \frac{4^\alpha}{e^R} \int_{-T}^{T} \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha)\Gamma(n+1)} \left(1 - \frac{2}{e^{R+x/e^R}+1}\right)^{2n+1} \frac{e^{\alpha(R+x/e^R)}}{(e^{R+x/e^R}+1)^{2\alpha}} \mathrm{d}x \\
&\leq \frac{2^{2\alpha+1}T}{e^R} \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha)\Gamma(n+1)} \left(1 - \frac{2}{e^{R+T/e^R}+1}\right)^{2n+1} \frac{1}{e^{\alpha(R-T/e^R)}} \\
&\leq \frac{2^{4\alpha+1}T}{e^R} \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha)\Gamma(n+1)} \left(1 - \frac{1}{2e^R}\right)^{2n+1} \frac{1}{e^{\alpha R}}.
\end{aligned}$$

Notice that there exists a constant $C > 0$ depending only on $\alpha$ such that for any $n \in \mathbb{N}$,

$$\frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha)\Gamma(n+1)} \leq Cn^\alpha,$$

and

$$\left(1 - \frac{1}{2e^R}\right)^{2n+1} = \left[\left(1 + \frac{1}{2e^R - 1}\right)^{2e^R}\right]^{-\frac{2n+1}{2e^R}} \leq e^{-n/e^R}.$$

Hence

$$\mathbb{P}(X_n^{(R)} \in [-T, T]) \leq \frac{2^{4\alpha+1}CT}{e^R} \left(n/e^R\right)^\alpha e^{-n/e^R}.$$

Since $x^\alpha e^{-x}$ is bounded for $x > 0$, we get

$$\sup_{n\geq 0} \mathbb{P}\left(X_n^{(R)} \in [-T, T]\right) = O\left(e^{-R}\right).$$

This completes the proof of Theorem 1.7.

**6.3. Proof of Theorem 1.8 (Ginibre case).** Recall that $(\rho_n)_{n\geq 0}$ is a family of non-negative independent random variables such that

$$\rho_n \sim \frac{2r^{2n+1}e^{-r^2}}{n!} \mathrm{d}r.$$

If we define $X_n^{(R)} = R(\rho_n - R)$ for each $R > 0$ and $n \in \mathbb{N}$, we can obtain that

$$\{R(|z| - R) : z \in \mathcal{G}\} \sim \left\{X_n^{(R)} : n \in \mathbb{N}\right\}.$$

We are going to show that

$$\left\{X_n^{(R)} : n \in \mathbb{N}\right\} \xrightarrow[R\to+\infty]{\text{law}} \mathcal{P}_2.$$

For every measurable compactly supported function $f:\mathbb{R}\to[0,1)$, a similar argument as Section 5 gives that

$$\sum_{n=0}^{\infty}\mathbb{E}\left[f(X_n^{(R)})\right]=2\int_{-R^2}^{+\infty}f(x)\mathrm{d}x+\frac{2}{R^2}\int_{-R^2}^{+\infty}xf(x)\,\mathrm{d}x\xrightarrow[R\to+\infty]{}2\int_{\mathbb{R}}f(x)\,\mathrm{d}x.$$

Hence by Corollary 6.2, we shall prove that for every $T>0$,

$$\sup_{n\geq 0}\mathbb{P}\left(X_n^{(R)}\in[-T,T]\right)\xrightarrow[R\to+\infty]{}0.$$

When $R$ is large enough, we have

$$\begin{aligned}\mathbb{P}\left(X_n^{(R)}\in[-T,T]\right)&=\frac{2}{R}\int_{-T}^{T}\frac{(R+\frac{x}{R})^{2n+1}}{n!}e^{-(R+\frac{x}{R})^2}\,\mathrm{d}x\\&\leq 4\frac{R^{2n}\left(1+\frac{T}{R^2}\right)^{2n}}{n!}e^{-R^2}\int_{-T}^{T}e^{-2x-\frac{x^2}{R^2}}\,\mathrm{d}x\\&\leq 8Te^{2T}\frac{R^{2n}e^{\frac{2nT}{R^2}}}{n!}e^{-R^2}\\&\leq 8Te^{2T}\frac{\left(\left\lceil R^2e^{\frac{2T}{R^2}}\right\rceil\right)^n}{n!}e^{-R^2}.\end{aligned}$$

Notice that

$$\frac{\left\lceil R^2e^{\frac{2T}{R^2}}\right\rceil^n}{n!}\leq\frac{\left(\left\lceil R^2e^{\frac{2T}{R^2}}\right\rceil\right)^{\left\lceil R^2e^{\frac{2T}{R^2}}\right\rceil}}{\left\lceil R^2e^{\frac{2T}{R^2}}\right\rceil!}=O\left(\frac{e^{\left\lceil R^2e^{\frac{2T}{R^2}}\right\rceil}}{\sqrt{\left\lceil R^2e^{\frac{2T}{R^2}}\right\rceil}}\right)=O\left(\frac{e^{R^2}}{R}\right),$$

where we used the Stirling's approximation. It follows that

$$\sup_{n\geq 0}\mathbb{P}\left(X_n^{(R)}\in[-T,T]\right)=O(R^{-1}).$$

This completes the proof of Theorem 1.8.

## 7. The case of superexponential growth

**7.1. Fluctuations in the hyperbolic case.** Theorem 1.7 explains that the limiting behavior of the point process $\{e^R(|z|_{\mathrm{h}}-R):z\in\mathcal{H}_\alpha\}$ is Poisson when $R\to+\infty$. For a bounded measurable compactly supported function $f$ on $\mathbb{R}$, Theorem 1.3 tells that the limiting behavior of $\sum_{z\in\mathcal{H}_\alpha}f(|z|_{\mathrm{h}}-R)$ is Gaussian when $R\to+\infty$. In the case $1\ll a_R\ll e^R$ as $R\to+\infty$, Theorem 1.5 shows that the limiting behavior of $\sum_{z\in\mathcal{H}_\alpha}f\big(a_R(|z|_{\mathrm{h}}-R)\big)$ is also Gaussian. For completeness, we continue to consider the limiting behavior of $\sum_{z\in\mathcal{H}_\alpha}f\big(a_R(|z|_{\mathrm{h}}-R)\big)$ in the case $a_R\gg e^R$ and $a_R\ll 1$ as $R\to+\infty$.

In the case $a_R \gg e^R$ as $R \to +\infty$, for a bounded measurable and compactly supported function $f : \mathbb{R} \to \mathbb{R}$, the limiting behavior of $\sum_{z\in\mathcal{H}_\alpha} f\big(a_R(|z|_\mathrm{h} - R)\big)$ is zero. This can be seen by

$$\operatorname{Var}\left(\sum_{z\in\mathcal{H}_\alpha} f\big(a_R(|z|_\mathrm{h} - R)\big)\right) \xrightarrow[R\to+\infty]{} 0.$$

In fact, a similar argument as in Section 4 implies that

$$\begin{aligned}\operatorname{Var}\left(\sum_{z\in\mathcal{H}_\alpha} f\big(a_R(|z|_\mathrm{h} - R)\big)\right) &\le \sum_{z\in\mathcal{H}_\alpha} \mathbb{E}\left[f^2\big(a_R(|z|_\mathrm{h} - R)\big)\right] \\ &\sim \frac{\alpha e^R}{4a_R}\int_{\mathbb{R}} f^2(x)\,\mathrm{d}x \xrightarrow[R\to+\infty]{} 0.\end{aligned}$$

We now turn to consider the case $a_R \ll 1$ as $R \to +\infty$. For a bounded measurable and compactly supported function $f : \mathbb{R} \to \mathbb{R}$, denote $M_f$ the essential right endpoint of $\operatorname{supp} f$, that is,

$$M_f := \inf\big\{M \in \mathbb{R} : f(x) = 0 \text{ almost everywhere on } [M, +\infty)\big\}.$$

**Theorem 7.1.** *Suppose that $a_R > 0$ satisfies $a_R \ll 1$ as $R \to +\infty$. Let $f$ be a real-valued bounded measurable function on $\mathbb{R}$ with compact support such that $f(M_f^-) := \lim_{x\to M_f^-} f(x)$ exists and is non-zero. If $R + M_f/a_R \to +\infty$ as $R \to +\infty$, then for each $\alpha > 0$,*

$$\frac{\sum_{z\in\mathcal{H}_\alpha} f\big(a_R(|z|_\mathrm{h} - R)\big) - \mathbb{E}\left[\sum_{z\in\mathcal{H}_\alpha} f\big(a_R(|z|_\mathrm{h} - R)\big)\right]}{\sqrt{\operatorname{Var}\left(\sum_{z\in\mathcal{H}_\alpha} f\big(a_R(|z|_\mathrm{h} - R)\big)\right)}} \xrightarrow[R\to+\infty]{\text{law}} \mathcal{N}(0,1).$$

With the help of Theorem 7.1, we can do a more detailed discussion when $a_R \ll 1$ as $R \to +\infty$.

(i) In the case $R^{-1} \ll a_R \ll 1$, we always have $R + M_f/a_R \to +\infty$, so the limiting behavior of $\sum_{z\in\mathcal{H}_\alpha} f\big(a_R(|z|_\mathrm{h} - R)\big)$ is Gaussian when $f(M_f^-)$ exists and is non-zero.

(ii) In the case $a_R = R^{-1}$, if $M_f \le -1$, since $a_R(|z|_\mathrm{h} - R) > -1$ except for $z = 0$, $\sum_{z\in\mathcal{H}_\alpha} f\big(R^{-1}(|z|_\mathrm{h} - R)\big)$ is almost surely the zero random variable for every $R > 0$; if $M_f > -1$, we have $R + M_f/a_R \to +\infty$, so the limiting behavior of $\sum_{z\in\mathcal{H}_\alpha} f\big(R^{-1}(|z|_\mathrm{h} - R)\big)$ is Gaussian when $f(M_f^-)$ exists and is non-zero.

(iii) In the case $a_R \ll R^{-1}$, if $M_f < 0$, since $a_R(|z|_\mathrm{h} - R) > -Ra_R \to 0$ except $z = 0$, $\sum_{z\in\mathcal{H}_\alpha} f\big(a_R(|z|_\mathrm{h} - R)\big)$ is almost surely the zero random variable for sufficiently large $R$; if $M_f \ge 0$, we have $R + M_f/a_R \to +\infty$, so the limiting behavior of $\sum_{z\in\mathcal{H}_\alpha} f\big(a_R(|z|_\mathrm{h} - R)\big)$ is Gaussian when $f(M_f^-)$ exists and is non-zero.

*Question* 1. For the hyperbolic situation, in the case $a_R = R^{-1}$ and $M_f > -1$, or $a_R \ll R^{-1}$ and $M_f \ge 0$, does the central limit theorem also holds without the condition that $f(M_f^-)$ exists and is non-zero?

*Proof of Theorem 7.1.* We will use Soshnikov's Theorem 2.3 to prove this theorem. The calculations are similar as Section 4.

Let us first calculate the expectation

$$\mathbb{E}\left[\sum_{z\in\mathcal{H}_\alpha}\left|f\left(a_R(|z|_{\mathrm{h}}-R)\right)\right|\right]=\frac{4^\alpha e^R}{a_R}\int_{-Ra_R}^{M_f}\int_0^{+\infty}|f(x)|e^{\frac{x}{a_R}}k^{(\alpha)}_{\lfloor te^{R+\frac{x}{a_R}}\rfloor}$$
$$\times\left(1-\frac{2}{e^{R+\frac{x}{a_R}}+1}\right)^{2\left\lfloor te^{R+\frac{x}{a_R}}\right\rfloor+1}\frac{e^{\alpha\left(R+\frac{x}{a_R}\right)}}{\left(e^{R+\frac{x}{a_R}}+1\right)^{2\alpha}}\,\mathrm{d}t\,\mathrm{d}x.$$

Make a variable substitution by $x=a_Ry+M_f$, then

$$\mathbb{E}\left[\sum_{z\in\mathcal{H}_\alpha}\left|f\left(a_R(|z|_{\mathrm{h}}-R)\right)\right|\right]$$
$$=4^\alpha e^{R+M_f/a_R}\int_{-R-M_f/a_R}^{0}\int_0^{+\infty}|f(a_Ry+M_f)|\,e^yk^{(\alpha)}_{\left\lfloor te^{R+M_f/a_R+y}\right\rfloor}$$
$$\times\left(1-\frac{2}{e^{R+M_f/a_R+y}+1}\right)^{2\left\lfloor te^{R+M_f/a_R+y}\right\rfloor+1}\frac{e^{\alpha(R+M_f/a_R+y)}}{\left(e^{R+M_f/a_R+y}+1\right)^{2\alpha}}\,\mathrm{d}t\,\mathrm{d}y.$$

Hence

$$\mathbb{E}\left[\sum_{z\in\mathcal{H}_\alpha}\left|f\left(a_R(|z|_{\mathrm{h}}-R)\right)\right|\right]\sim\frac{\alpha|f(M_f^-)|}{4}e^{R+M_f/a_R}.$$

Next we shall calculate the variance

$$\mathrm{Var}\left(\sum_{z\in\mathcal{H}_\alpha}f\left(a_R(|z|_{\mathrm{h}}-R)\right)\right)$$
$$=\sum_{n=0}^{\infty}\mathbb{E}\left[f^2\left(a_R(|\rho_n^{(\alpha)}|_{\mathrm{h}}-R)\right)\right]-\sum_{n=0}^{\infty}\mathbb{E}\left[f\left(a_R(|\rho_n^{(\alpha)}|_{\mathrm{h}}-R)\right)\right]^2.$$

For the first term, we have

$$\sum_{n=0}^{\infty}\mathbb{E}\left[f^2\left(a_R(|\rho_n^{(\alpha)}|_{\mathrm{h}}-R)\right)\right]\sim\frac{\alpha|f(M_f^-)|^2}{4}e^{R+M_f/a_R}.$$

As for the second term

$$\sum_{n=0}^{\infty}\mathbb{E}\left[f\left(a_R(|\rho_n^{(\alpha)}|_{\mathrm{h}}-R)\right)\right]^2$$
$$=\frac{4^{2\alpha}}{a_R^2}\sum_{n=0}^{\infty}\left(\int_{-Ra_R}^{M_f}f(x)k_n^{(\alpha)}\left(1-\frac{2}{e^{R+\frac{x}{a_R}}+1}\right)^{2n+1}\frac{e^{\alpha\left(R+\frac{x}{a_R}\right)}}{\left(e^{R+\frac{x}{a_R}}+1\right)^{2\alpha}}\,\mathrm{d}x\right)^2,$$

set $n = \lfloor te^{R+M_f/a_R} \rfloor$, then

$$\begin{aligned}
&\sum_{n=0}^{\infty} \mathbb{E}\Big[f\big(a_R(|\rho_n^{(\alpha)}|_{\mathrm{h}} - R)\big)\Big]^2 \\
&\quad = 4^{2\alpha} \frac{e^{R+M_f/a_R}}{a_R^2} \int_0^{+\infty} \Bigg( \int_{-Ra_R}^{M_f} f(x) k^{(\alpha)}_{\lfloor te^{R+M_f/a_R} \rfloor} \\
&\qquad \times \left(1 - \frac{2}{e^{R+\frac{x}{a_R}}+1}\right)^{2\lfloor te^{R+M_f/a_R}\rfloor+1} \frac{e^{\alpha(R+\frac{x}{a_R})}}{(e^{R+\frac{x}{a_R}}+1)^{2\alpha}} \mathrm{d}x \Bigg)^2 \mathrm{d}t.
\end{aligned}$$

Make a variable substitution by $x = a_R y + M_f$, so

$$\begin{aligned}
&\sum_{n=0}^{\infty} \mathbb{E}\Big[f\big(a_R(|\rho_n^{(\alpha)}|_{\mathrm{h}} - R)\big)\Big]^2 \\
&\quad = 4^{2\alpha} e^{R+M_f/a_R} \int_0^{+\infty} \Bigg( \int_{-R-M_f/a_R}^{0} f(a_R y + M_f) k^{(\alpha)}_{\lfloor te^{R+M_f/a_R} \rfloor} \\
&\qquad \times \left(1 - \frac{2}{e^{R+M_f/a_R+y}+1}\right)^{2\lfloor te^{R+M_f/a_R}\rfloor+1} \frac{e^{\alpha(R+M_f/a_R+y)}}{\left(e^{R+M_f/a_R+y}+1\right)^{2\alpha}} \mathrm{d}y \Bigg)^2 \mathrm{d}t \\
&\quad \sim e^{R+M_f/a_R} \frac{\alpha |f(M_f^-)|^2}{4B(\alpha,\alpha+1)} \int_{-\infty}^{0}\int_{-\infty}^{0} \frac{e^{(\alpha+1)(x+y)}}{(e^x+e^y)^{2\alpha+1}} \,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Notice that

$$\frac{1}{B(\alpha,\alpha+1)} \int_{-\infty}^{+\infty}\int_{-\infty}^{0} \frac{e^{(\alpha+1)(x+y)}}{(e^x+e^y)^{2\alpha+1}} \,\mathrm{d}x\,\mathrm{d}y = 1,$$

we conclude that

$$\begin{aligned}
&\mathrm{Var}\left( \sum_{z\in\mathcal{H}_\alpha} f\big(a_R(|z|_{\mathrm{h}} - R)\big) \right) \\
&\qquad \sim e^{R+M_f/a_R} \frac{\alpha |f(M_f^-)|^2}{4B(\alpha,\alpha+1)} \int_0^{+\infty}\int_{-\infty}^{0} \frac{e^{(\alpha+1)(x+y)}}{(e^x+e^y)^{2\alpha+1}} \mathrm{d}x\mathrm{d}y.
\end{aligned}$$

The above calculations yield that when $R \to +\infty$,

$$\mathbb{E}\left[ \sum_{z\in\mathcal{H}_\alpha} \left| f\big(a_R(|z|_{\mathrm{h}} - R)\big) \right| \right] = O\left( \mathrm{Var}\left( \sum_{z\in\mathcal{H}_\alpha} f\big(a_R(|z|_{\mathrm{h}} - R)\big) \right) \right),$$

and then we can use Soshnikov's Theorem 2.3 directly to get the central limit theorem.

This completes the proof of Theorem 7.1. □

**7.2. Fluctuations in the Ginibre case.** Theorem 1.8 explains that the limiting behavior of the point process $\{R(|z|-R) : z \in \mathcal{G}\}$ is Poisson when $R \to +\infty$. For a bounded measurable compactly supported function $f$ on $\mathbb{R}$, Theorem 1.4 illustrates that the limiting behavior of $\sum_{z\in\mathcal{G}} f(|z|-R)$ is Gaussian when $R \to +\infty$. In the case $1 \ll a_R \ll R$ as $R \to +\infty$, Theorem 1.6 shows that the limiting behavior of $\sum_{z\in\mathcal{G}} f\big(a_R(|z|-R)\big)$ is also Gaussian. For completeness, we continue to consider the limiting behavior of $\sum_{z\in\mathcal{G}} f\big(a_R(|z|-R)\big)$ in the case $a_R \gg R$ and $a_R \ll 1$ as $R \to +\infty$.

In the case $a_R \gg R$ as $R \to +\infty$, for a bounded measurable and compactly supported function $f : \mathbb{R} \to \mathbb{R}$, the limiting behavior of $\sum_{z\in\mathcal{G}} f\big(a_R(|z|-R)\big)$ is zero. This can be seen by

$$\operatorname{Var}\left(\sum_{z\in\mathcal{G}} f\big(a_R(|z|-R)\big)\right) \xrightarrow[R\to+\infty]{} 0.$$

In fact, a similar argument as Section 5 implies that

$$\begin{aligned}\operatorname{Var}\left(\sum_{z\in\mathcal{G}} f\big(a_R(|z|-R)\big)\right) &\le \sum_{z\in\mathcal{G}} \mathbb{E}\left[f^2\big(a_R(|z|-R)\big)\right] \\ &\sim \frac{2R}{a_R}\int_{\mathbb{R}} f^2(x)\,\mathrm{d}x \xrightarrow[R\to+\infty]{} 0.\end{aligned}$$

We now turn to consider the case $a_R \ll 1$ as $R \to +\infty$.

**Theorem 7.2.** *Suppose that $a_R > 0$ satisfies $a_R \ll 1$ as $R \to +\infty$. Let $f$ be a real-valued bounded measurable function on $\mathbb{R}$ with compact support such that $f(M_f^-) := \lim_{x\to M_f^-} f(x)$ exists and is non-zero. If $R + M_f/a_R \to +\infty$ as $R \to +\infty$, then,*

$$\frac{\sum_{z\in\mathcal{G}} f\big(a_R(|z|-R)\big) - \mathbb{E}\big[\sum_{z\in\mathcal{G}} f\big(a_R(|z|-R)\big)\big]}{\sqrt{\operatorname{Var}\big(\sum_{z\in\mathcal{G}} f\big(a_R(|z|-R)\big)\big)}} \xrightarrow[R\to+\infty]{\text{law}} \mathcal{N}(0,1).$$

With the help of Theorem 7.2, we can do a more detailed discussion when $a_R \ll 1$ as $R \to +\infty$.

(i) In the case $R^{-1} \ll a_R \ll 1$, we always have $R + M_f/a_R \to +\infty$, so the limiting behavior of $\sum_{z\in\mathcal{G}} f\big(a_R(|z|-R)\big)$ is Gaussian when $f(M_f^-)$ exists and is non-zero.

(ii) In the case $a_R = R^{-1}$, if $M_f \le -1$, since $a_R(|z|-R) > -1$ except for $z = 0$, $\sum_{z\in\mathcal{G}} f\big(R^{-1}(|z|-R)\big)$ is almost surely the zero random variable for every $R > 0$; if $M_f > -1$, we have $R + M_f/a_R \to +\infty$, so the limiting behavior of $\sum_{z\in\mathcal{G}} f\big(R^{-1}(|z|-R)\big)$ is Gaussian when $f(M_f^-)$ exists and is non-zero.

(iii) In the case $a_R \ll R^{-1}$, if $M_f < 0$, since $a_R(|z|-R) > -Ra_R \to 0$ except $z = 0$, $\sum_{z\in\mathcal{G}} f\big(a_R(|z|-R)\big)$ is almost surely the zero random variable for sufficiently large $R$; if $M_f \ge 0$, we have $R + M_f/a_R \to +\infty$, so the limiting behavior of $\sum_{z\in\mathcal{G}} f\big(a_R(|z|-R)\big)$ is Gaussian when $f(M_f^-)$ exists and is non-zero.

*Question* 2. For the Ginibre situation, in the case $a_R = R^{-1}$ and $M_f > -1$, or $a_R \ll R^{-1}$ and $M_f \geq 0$, does the central limit theorem also holds without the condition that $f(M_f^-)$ exists and is non-zero?

*Proof of Theorem 7.2.* We will use Soshnikov's Theorem 2.3 to prove this theorem. The calculations are similar as Section 5.

Let us first calculate the expectation

$$\mathbb{E}\left[\sum_{z\in\mathcal{G}} \left|f\big(a_R(|z|-R)\big)\right|\right] = \frac{2R}{a_R}\int_{-Ra_R}^{M_f} |f(x)|\,\mathrm{d}x + \frac{2}{a_R^2}\int_{-Ra_R}^{M_f} x|f(x)|\,\mathrm{d}x = \frac{2}{a_R}\int_{-Ra_R}^{M_f} |f(x)|\left(R+\frac{x}{a_R}\right)\mathrm{d}x.$$

Make a variable substitution by $x = a_R y + M_f$, then

$$\mathbb{E}\left[\sum_{z\in\mathcal{G}} \left|f\big(a_R(|z|-R)\big)\right|\right] = 2\int_{-R-M_f/a_R}^{0} |f(a_R y + M_f)|\big(R + M_f/a_R + y\big)\,\mathrm{d}y \leq 2\,\|f\|_\infty\,(R+M_f/a_R)^2.$$

Next we shall calculate the variance

$$\begin{aligned}&\operatorname{Var}\left(\sum_{z\in\mathcal{G}} f\big(a_R(|z|-R)\big)\right)\\ &= \frac{2}{\pi a_R^2}\int_{-Ra_R}^{+\infty}\int_{-Ra_R}^{+\infty}\int_0^{\pi\sqrt{\left(R+\frac{x}{a_R}\right)\left(R+\frac{y}{a_R}\right)}} \big[f(x)-f(y)\big]^2 e^{-\frac{(x-y)^2}{a_R^2}}\\ &\quad\times\sqrt{\left(R+\frac{x}{a_R}\right)\left(R+\frac{y}{a_R}\right)}\\ &\quad\times e^{-4\left(R+\frac{x}{a_R}\right)\left(R+\frac{y}{a_R}\right)\sin^2\frac{t}{2\sqrt{\left(R+\frac{x}{a_R}\right)\left(R+\frac{y}{a_R}\right)}}}\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y = I_1(R) - I_2(R),\end{aligned}$$

where

$$\begin{aligned}I_1(R) = \frac{4}{\pi a_R^2}&\int_{-Ra_R}^{+\infty}\int_{-Ra_R}^{M_f}\int_0^{\pi\sqrt{\left(R+\frac{x}{a_R}\right)\left(R+\frac{y}{a_R}\right)}} f^2(x) e^{-\frac{(x-y)^2}{a_R^2}}\\ &\times\sqrt{\left(R+\frac{x}{a_R}\right)\left(R+\frac{y}{a_R}\right)}\\ &\times e^{-4\left(R+\frac{x}{a_R}\right)\left(R+\frac{y}{a_R}\right)\sin^2\frac{t}{2\sqrt{\left(R+\frac{x}{a_R}\right)\left(R+\frac{y}{a_R}\right)}}}\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

and

$$I_2(R) = \frac{4}{\pi a_R^2}\int_{-Ra_R}^{M_f}\int_{-Ra_R}^{M_f}\int_0^{\pi\sqrt{\left(R+\frac{x}{a_R}\right)\left(R+\frac{y}{a_R}\right)}} f(x)f(y) e^{-\frac{(x-y)^2}{a_R^2}}$$

$$\times \sqrt{\left(R+\frac{x}{a_R}\right)\left(R+\frac{y}{a_R}\right)}$$
$$\times e^{-4\left(R+\frac{x}{a_R}\right)\left(R+\frac{y}{a_R}\right)\sin^2\frac{t}{2\sqrt{\left(R+\frac{x}{a_R}\right)\left(R+\frac{y}{a_R}\right)}}}\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y$$
$$\leq \frac{4}{\pi a_R^2}\int_{-Ra_R}^{M_f}\int_{-Ra_R}^{M_f}\int_0^{\pi\sqrt{\left(R+\frac{x}{a_R}\right)\left(R+\frac{y}{a_R}\right)}} f^2(x)e^{-\frac{(x-y)^2}{a_R^2}}$$
$$\times \sqrt{\left(R+\frac{x}{a_R}\right)\left(R+\frac{y}{a_R}\right)}$$
$$\times e^{-4\left(R+\frac{x}{a_R}\right)\left(R+\frac{y}{a_R}\right)\sin^2\frac{t}{2\sqrt{\left(R+\frac{x}{a_R}\right)\left(R+\frac{y}{a_R}\right)}}}\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y.$$

It follows that

$$\mathrm{Var}\left(\sum_{z\in\mathcal{G}} f\big(a_R(|z|-R)\big)\right)$$
$$\geq \frac{4}{\pi a_R^2}\int_{M_f}^{+\infty}\int_{-Ra_R}^{M_f}\int_0^{\pi\sqrt{\left(R+\frac{x}{a_R}\right)\left(R+\frac{y}{a_R}\right)}} f^2(x)e^{-\frac{(x-y)^2}{a_R^2}}$$
$$\times \sqrt{\left(R+\frac{x}{a_R}\right)\left(R+\frac{y}{a_R}\right)}$$
$$\times e^{-4\left(R+\frac{x}{a_R}\right)\left(R+\frac{y}{a_R}\right)\sin^2\frac{t}{2\sqrt{\left(R+\frac{x}{a_R}\right)\left(R+\frac{y}{a_R}\right)}}}\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y.$$

Set $x = a_R u + M_f$ and $y = a_R v + M_f$, then

$$\mathrm{Var}\left(\sum_{z\in\mathcal{G}} f\big(a_R(|z|-R)\big)\right)$$
$$\geq \frac{4}{\pi}\int_0^{+\infty}\int_{-R-M_f/a_R}^{0}\int_0^{\pi\sqrt{(R+M_f/a_R+u)(R+M_f/a_R+v)}} f^2(a_Ru+M_f)e^{-(u-v)^2}$$
$$\times \sqrt{(R+M_f/a_R+u)(R+M_f/a_R+v)}$$
$$\times e^{-4(R+M_f/a_R+u)(R+M_f/a_R+v)\sin^2\frac{t}{2\sqrt{(R+M_f/a_R+u)(R+M_f/a_R+v)}}}\,\mathrm{d}t\,\mathrm{d}u\,\mathrm{d}v$$
$$\sim \frac{2}{\sqrt{\pi}}|f(M_f^-)|^2(R+M_f/a_R)\int_0^{+\infty}\int_{-\infty}^{0} e^{-(u-v)^2}\,\mathrm{d}u\,\mathrm{d}v.$$

The above calculations yield that when $R\to+\infty$,

$$\mathbb{E}\left[\sum_{z\in\mathcal{G}}\left|f\big(a_R(|z|-R)\big)\right|\right] = O\left(\left(\mathrm{Var}\left(\sum_{z\in\mathcal{G}} f\big(a_R(|z|-R)\big)\right)\right)^2\right),$$

and then we can use Soshnikov's Theorem 2.3 directly to get the central limit theorem.

This completes the proof of Theorem 7.2. □

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## References

[1] A. Borodin, *Determinantal point processes,* Oxford Handbook of Random Matrix Theory, Oxford Univ. Press, Oxford, 2011.

[2] A.I. Bufetov, *The conditional measures for the determinantal point process with the Bergman kernel,* preprint, https://arxiv.org/abs/2112.15557.

[3] A.I. Bufetov and A.V. Dymov, *A functional limit theorem for the sine-process,* Int. Math. Res. Not. IMRN (2019), No. 1, 249–319.

[4] A.I. Bufetov and Y. Qiu, *Determinantal point processes associated with Hilbert spaces of holomorphic functions,* Comm. Math. Phys. **351** (2017), 1–44.

[5] D.J. Daley and D. Vere-Jones, *An introduction to the theory of point processes. Vol. II,* General theory and structure. Second edition. Probability and its Applications (New York), Springer, New York, 2008.

[6] S. Ghosh, *Determinantal processes and completeness of random exponentials: the critical case,* Probab. Theory Related Fields **163** (2015), 643–665.

[7] S. Ghosh and Y. Peres, *Rigidity and tolerance in point processes: Gaussian zeros and Ginibre eigenvalues,* Duke Math. J. **166** (2017), 1789–1858.

[8] J. Ginibre, *Statistical ensembles of complex, quaternion, and real matrices,* J. Mathematical Phys. **6** (1965), 440–449.

[9] J.B. Hough, M. Krishnapur, Y. Peres, and B. Virág, *Zeros of Gaussian analytic functions and determinantal point processes,* University Lecture Series, 51. American Mathematical Society, Providence, RI, 2009.

[10] Y.G. Kondratiev and T. Kuna, *Harmonic analysis on configuration space. I. General theory,* Infin. Dimens. Anal. Quantum Probab. Relat. Top. **5** (2002), 201–233.

[11] E. Kostlan, *On the spectra of Gaussian matrices,* Linear Algebra Appl. **162/164** (1992), 385–388.

[12] M. Krishnapur, *From random matrices to random analytic functions,* Ann. Probab. **37** (2009), 314–346.

[13] A. Lenard, *Correlation functions and the uniqueness of the state in classical statistical mechanics,* Comm. Math. Phys. **30** (1973), 35–44.

[14] O. Macchi, *The coincidence approach to stochastic point processes*, Advances in Appl. Probability. **7** (1975), 83–122.

[15] Y. Peres and B. Virág, *Zeros of the i.i.d. Gaussian power series: A conformally invariant determinantal process*, Acta Math. **194** (2005), 1–35.

[16] T. Shirai and Y. Takahashi, *Fermion process and Fredholm determinant*, Proceedings of the Second ISAAC Congress, Vol. 1 (Fukuoka, 1999). Int. Soc. Anal. Appl. Comput., 7, Kluwer Acad. Publ., Dordrecht, 2000, 15–23.

[17] A. Soshnikov, *Determinantal random point fields*, Russian Math. Surveys. **55** (2000), 923–975.

[18] A. Soshnikov, *Gaussian limit for determinantal random point fields*, Ann. Probab. **30** (2002), 171–187.

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## Флуктуації процесу модулей для гіперболічного ансамблю та ансамблю Жінібра

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Ми досліджуємо точковий процес модулів ансамблю Жінібра та гіперболічного ансамблю. Ми доводимо, що вдалині від початку координат і відносно певної шкали ці процеси виявляють пуассонові і гаусові флуктуації. Серед можливих гаусових флуктуацій ми можемо знайти білий шум, а також гаусові флуктуації з нетривіальною коваріацією на деяких шкалах.

*Ключові слова:* ансамбль Жінібра, гіперболічний ансамбль, процес модулей, нормальний розподіл, білий шум, пуассонів точковий процес